

# Multi-User Scheduling in the 3GPP LTE Cellular Uplink

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## Abstract

In this paper, we consider resource allocation in the 3GPP Long Term Evolution (LTE) cellular uplink, which will be the most widely deployed next generation cellular uplink. The key features of the 3GPP LTE uplink (UL) are that it is based on a modified form of the orthogonal frequency division multiplexing based multiple access (OFDMA) which enables channel dependent frequency selective scheduling, and that it allows for multi-user (MU) scheduling wherein multiple users can be assigned the same time-frequency resource. In addition to the considerable spectral efficiency improvements that are possible by exploiting these two features, the LTE UL allows for transmit antenna selection together with the possibility to employ advanced receivers at the base-station, which promise further gains. However, several practical constraints that seek to maintain a low signaling overhead, are also imposed. In this paper, we show that the resulting resource allocation problem is APX-hard and then propose a *local ratio test (LRT)* based constant-factor polynomial-time approximation algorithm. We then propose two enhancements to this algorithm as well as a sequential LRT based MU scheduling algorithm that offers a constant-factor approximation and is another useful choice in the complexity versus performance tradeoff. Further, user pre-selection, wherein a smaller pool of *good* users is pre-selected and a sophisticated scheduling algorithm is then employed on the selected pool, is also examined. We suggest several such user pre-selection algorithms, some of which are shown to offer constant-factor approximations to the pre-selection problem. Detailed evaluations reveal that the proposed algorithms and their enhancements offer significant gains.

## I. INTRODUCTION

The next generation cellular systems, a.k.a. 4G cellular systems, will operate over wideband multi-path fading channels and have chosen OFDMA as their air-interface [1]. The motivating factors behind the choice of OFDMA are that it is an effective means to handle multi-path fading and that it allows for enhancing multi-user diversity gains via channel-dependent frequency-domain scheduling. The deployment of 4G cellular systems has begun

and will accelerate in the coming years. Predominantly the 4G cellular systems will be based on the 3GPP LTE standard [1] since an overwhelming majority of cellular operators have committed to LTE and specifically all deployments in the foreseeable future will adhere to the first version of the LTE standard, referred to as Release 8. Our focus in this paper is on the uplink (UL) in these Release 8 LTE based cellular systems (henceforth referred to simply as LTE UL) and in particular on multi-user (MU) scheduling for the LTE UL. The LTE UL employs a modified form of OFDMA, referred to as the DFT-Spread-OFDMA [1]. In each scheduling interval, the available system bandwidth is partitioned into multiple resource blocks (RBs), where each RB represents the minimum allocation unit and is a pre-defined set of consecutive subcarriers and OFDM symbols. The scheduler is a frequency domain packet scheduler, which in each scheduling interval assigns these RBs to the individual users. Anticipating a rapid growth in data traffic, the LTE UL has enabled MU scheduling along with transmit antenna selection. Unlike single-user (SU) scheduling, a key feature of MU scheduling is that an RB can be simultaneously assigned to more than one user in the same scheduling interval. MU scheduling is well supported by fundamental capacity and degrees of freedom based analysis [2], [3] and indeed, its promised gains need to be harvested in order to cater to the ever increasing traffic demands. However, several constraints have also been placed by the LTE standard on such MU scheduling (and the resulting MU transmissions). These constraints seek to balance the need to provide scheduling freedom with the need to ensure a low signaling overhead and respect device limitations. The design of an efficient and implementable MU scheduler for the LTE UL is thus an important problem.

In Fig. 1 we highlight the key constraints in LTE MU scheduling by depicting a feasible allocation. Notice first that all RBs assigned to a user must form a chunk of contiguous RBs and each user can be assigned at-most one such chunk. This restriction allows us to exploit frequency domain channel variations via localized assignments (there is complete freedom in choosing the location and size of each such chunk) while respecting strict limits on the per-user transmit peak-to-average-power-ratio (PAPR). Note also that there should be a complete overlap among any two users that share an RB. In other words, if any two users are co-scheduled on an RB then those two users must be co-scheduled on all their assigned RBs. This constraint is a consequence of Zadoff-Chu (ZC) sequences (and their cyclic shifts) being used as pilot sequences in the LTE UL [1] and is needed to ensure reliable channel estimation. The LTE UL further assumes that each user can have multiple transmit antennas but is equipped with only one power amplifier due to cost constraints. Accordingly, it allows a basic precoding in the form of transmit antenna selection where each scheduled user can be informed about the transmit antenna it should employ in a scheduling interval. In addition, to minimize the signaling overhead, each scheduled user

can transmit with only one power level (or power spectral density (PSD)) on all its assigned RBs. This PSD is implicitly determined by the number of RBs assigned to that user, i.e., the user divides its total power equally among all its assigned RBs subject possibly to a spectral mask constraint. While this constraint significantly decreases the signaling overhead involved in conveying the scheduling decisions to the users, it does not result in any significant performance degradation. This is due to the fact that the multi-user diversity effect ensures that each user is scheduled on the set of RBs on which it has relatively good channels. A constant power allocation over such *good* channels results in a negligible loss [4]. Finally, scheduling in LTE UL must respect control channel overhead constraints and interference limit constraints. The former constraints arise because the scheduling decisions are conveyed to the users on the downlink control channel, whose limited capacity in turn places a limit on the set of users that can be scheduled. The latter constraints are employed to mitigate intercell interference. In the sequel it is shown that both these types of constraints can be posed as column-sparse and generic knapsack (linear packing) constraints, respectively.

The goal of this work is to design practical MU resource allocation algorithms for the LTE cellular uplink, where the term resource refers to RBs, modulation and coding schemes (MCS), power levels as well as choice of transmit antennas. In particular, we consider the design of resource allocation algorithms via weighted sum rate utility maximization, which accounts for finite user queues (buffers) and practical MCS. In addition, the designed algorithms comply with all the aforementioned practical constraints. Our main contributions are as follows:

- 1) We show that while the *complete* overlap constraint along with the at-most one chunk per scheduled user constraint make the resource allocation problem APX-hard, they greatly facilitate the use of local ratio test (LRT) based methods [5], [6]. We then design an LRT based polynomial time deterministic constant-factor approximation algorithm. A remarkable feature of this LRT based algorithm is that it is an end-to-end solution which can accommodate all constraints.
- 2) We then propose an enhancement that can significantly reduce the complexity of the LRT based MU scheduling algorithm while offering identical performance, as well as an enhancement that can yield good performance improvements with a very small additional complexity.
- 3) We propose a sequential LRT based MU scheduling algorithm that offers another useful choice in the complexity versus performance tradeoff. This algorithm also offers constant-factor approximation (albeit with a poorer constant) and a significantly reduced complexity.
- 4) In a practical system, it is useful to first pre-select a smaller pool of *good* users and then employ a sophisticated scheduling algorithm on the selected pool. Pre-selection can substantially reduce complexity

and is also a simple way to enforce a constraint on the number of users that can be scheduled in a scheduling interval. We note that another way to enforce the latter constraint is via a knapsack constraint in the LRT based MU scheduling. We suggest several such user pre-selection algorithms, some of which are shown to offer constant-factor approximations to the pre-selection problem.

- 5) The performance of the proposed LRT based MU scheduling algorithm together with its enhancements, the sequential LRT based MU scheduling algorithm and the proposed user pre-selection algorithms are evaluated for different BS receiver options via elaborate system level simulations that fully conform to the 3GPP evaluation methodology. It is seen that the proposed LRT based MU scheduling algorithm along with an advanced BS receiver can yield over 27% improvement in cell average throughput along with over 10% cell edge throughput improvement compared to SU scheduling. Its sequential counterpart is also attractive in that it yields about 20% improvement in cell average throughput while retaining the cell edge performance of SU scheduling. Further, it is seen that user pre-selection is indeed an effective approach and the suggested pre-selection approaches can offer significant gains.

#### A. Related Work

Resource allocation for the OFDM/OFDMA networks has been the subject of intense research [7]–[12]. A majority of OFDMA resource allocation problems hitherto considered belong to the class of single-user (SU) scheduling problems, which attempt to maximize a system utility by assigning non-overlapping subcarriers to users, along with transmit power levels for the assigned subcarriers. Even within this class most of the focus has been on the downlink. These resource allocation problems have been formulated as *continuous optimization problems*, which are in general non-linear and non-convex. As a result several approaches based on the game theory [13], [14], dual decomposition [7] or the analysis of optimality conditions [15] have been developed. Recent works have focused on the downlink in emerging cellular standards and have proposed approximation algorithms after modeling the resource allocation problems as constrained integer programs. Prominent examples are [10], [16] which consider the design of downlink SU-MIMO schedulers for LTE cellular systems and derive constant factor approximation algorithms.

Resource allocation for the DFT-Spread-OFDMA uplink has been relatively less studied with [6], [17]–[21] being the recent examples. In particular, [20] first considers a relaxed SU scheduling problem (without the integer valued RB allocation and the contiguity constraints) and poses the resource allocation problem as a convex optimization problem. It then proposes a fast interior point based method to solve that problem followed by a modification step to ensure contiguous allocation. A similar approach was adopted earlier in [22] where the

formulated convex optimization problem was solved via a sub-gradient method followed by a modification step to ensure integer valued RB allocation. Furthermore, [21] explicitly enforced the integer valued RB allocation constraint while formulating the resource allocation problem but also assumed that the chunk size for each user is given as an input, and proposed message passing based algorithms. Message passing based algorithms were also applied in [11] over an OFDMA uplink in order to minimize the total transmit power subject to rate guarantees. We note that while the algorithms in [20]–[22] may yield effective solutions in different regimes, they do not offer a worst-case performance guarantee and hence cannot be claimed to be approximation algorithms.

On the other hand, [6], [17]–[19] have explicitly modeled both integer valued RB allocation and the contiguity constraints. Specifically, [17] shows that the SU LTE UL scheduling problem is APX-hard and both [6], [17] provide deterministic constant-factor approximation algorithms, whereas [18] provides a randomized constant-factor approximation algorithm. [19] extends the algorithms of [6], [17] to the SU-MIMO LTE-A scheduling. The algorithm proposed in [6] is based on an innovative application of the LRT technique, which was developed earlier in [5]. However, we emphasize that the algorithms in [6], [17]–[19] cannot incorporate MU scheduling, do not consider user pre-selection and also cannot incorporate knapsack constraints. To the best of our knowledge the design of approximation algorithms for MU scheduling in the LTE uplink has not been considered before.

## II. MU SCHEDULING IN THE LTE UL

Consider a single-cell with  $K$  users and one BS which is assumed to have  $N_r \geq 1$  receive antennas. Suppose that user  $k$  has  $N_t \geq 1$  transmit antennas and its power budget is  $P_k$ . We let  $N$  denote the total number of RBs.

We consider the problem of scheduling users in the frequency domain in a given scheduling interval. Let  $\alpha_k$ ,  $1 \leq k \leq K$  denote the weight of the  $k^{th}$  user which is an input to the scheduling algorithm and is updated using the output of the scheduling algorithm in every scheduling interval, say according to the proportional fairness rule [23]. Letting  $r_k$  denote the rate assigned to the  $k^{th}$  user (in bits per N RBs), we consider the following weighted sum rate utility maximization problem,

$$\max \sum_{1 \leq k \leq K} \alpha_k r_k, \quad (1)$$

where the maximization is over the assignment of resources to the users **subject to:**

- **Decodability constraint:** The rates assigned to the scheduled users should be decodable by the base-station receiver. Notice that unlike SU scheduling, MU scheduling allows for multiple users to be assigned the same RB. As a result the rate that can be achieved for user  $k$  need not be only a function of the resources assigned to the  $k^{th}$  user but can also depend on the those assigned to the other users as well.

- **One transmit antenna and one power level per user:** Each user can transmit using only one power amplifier due to cost constraints. Thus, only a basic precoding in the form of transmit antenna selection is possible. In addition, each scheduled user is allowed to transmit with only one power level (or power spectral density (PSD)) on all its assigned RBs.
- **At most one chunk per-user and at-most  $T$  users per RB:** The set of RBs assigned to each scheduled user should form one chunk, where each chunk is a set of contiguous RBs. Further at-most  $T$  users can be co-scheduled on a given RB.  $T$  is expected to be small number typically two and no greater than four.
- **Complete overlap constraint:** If any two users are assigned a common RB then those two users must be assigned the same set of RBs. Feasible RB allocation and co-scheduling of users in LTE MU UL is depicted in Fig 1.
- **Finite buffers and finite MCS:** Users in a practical UL will have bursty traffic which necessitates considering finite buffers. In addition, only a finite set of MCS (29 possibilities in the LTE network) can be employed.
- **Control channel overhead constraints:** Every user that is given an UL grant (i.e., is scheduled on at least one RB) must be informed about its assigned MCS and the set of RBs on which it must transmit along with possibly the transmit antenna it should employ. This information is sent on the DL control channel of limited capacity which imposes a limit on the set of users that can be scheduled. In particular, the scheduling information of a user is encoded and formatted into one packet (henceforth referred to as a control packet), where the size of the control packet must be selected from a predetermined set of sizes. A longer (shorter) control packet is used for a cell edge (cell interior) user. In the LTE/LTE-A systems each user is assigned one search region when it enters the cell. In each scheduling interval it then searches for the control packet (containing the scheduling decisions made for it) only in that region of the downlink control channel, as well as a region common to all users.
- **Per sub-band interference limit constraints:** Inter-cell interference mitigation is performed by imposing interference limit constraints. In particular, on one or more subbands, the cell of interest must ensure that the total interference imposed by its scheduled users on a neighboring base-station is below a specified limit.

We define the set  $\mathcal{C}$  as the set containing  $N$  length vectors such that any  $\mathbf{c} \in \mathcal{C}$  is binary-valued with  $(\{0, 1\})$  elements and contains a contiguous sequence of ones with the remaining elements being zero. Here we say an RB  $i$  belongs to  $\mathbf{c}$  ( $i \in \mathbf{c}$ ) if  $\mathbf{c}$  contains a one in its  $i^{th}$  position, i.e.,  $c(i) = 1$ . Note then that each  $\mathbf{c} \in \mathcal{C}$  denotes a valid assignment of RBs since it contains one contiguous chunk of RBs. Also  $\mathbf{c}_1$  and  $\mathbf{c}_2$  are said to intersect if there is some RB that belongs to both  $\mathbf{c}_1$  and  $\mathbf{c}_2$ . For any  $\mathbf{c} \in \mathcal{C}$ , we will use  $\text{Tail}(\mathbf{c})$  ( $\text{Head}(\mathbf{c})$ ) to return the

largest (smallest) index that contains a one in  $\mathbf{c}$ . Thus, each  $\mathbf{c} \in \mathcal{C}$  has ones in all positions  $\text{Head}(\mathbf{c}), \dots, \text{Tail}(\mathbf{c})$  and zeros elsewhere. Further, we define  $\{\mathcal{G}_1, \dots, \mathcal{G}_L\}$  to be a partition of  $\{1, \dots, K\}$  with the understanding that all users that belong to a common set (or group)  $\mathcal{G}_s$ , for any  $1 \leq s \leq L$ , are mutually incompatible. In other words at-most one user from each group  $\mathcal{G}_s$  can be scheduled in a scheduling interval. Notice that by choosing  $L = K$  and  $\mathcal{G}_s = \{s\}$ ,  $1 \leq s \leq K$  we obtain the case where all users are mutually compatible. Let us define a family of subsets,  $\mathcal{U}$ , as

$$\mathcal{U} = \{\mathcal{U} \subseteq \{1, \dots, K\} : |\mathcal{U}| \leq T \text{ \& } |\mathcal{U} \cap \mathcal{G}_s| \leq 1 \forall 1 \leq s \leq L\} \quad (2)$$

and let  $\mathcal{M} = \mathcal{U} \times \mathcal{C}$ .

We can now pose the resource allocation problem as

$$\begin{aligned} & \max \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{M}} p(\mathcal{U}, \mathbf{c}) \mathcal{X}(\mathcal{U}, \mathbf{c}), \quad \text{s.t.} \\ & \text{For each group } \mathcal{G}_s, \quad \sum_{\substack{(\mathcal{U}, \mathbf{c}) \in \mathcal{M} \\ \mathcal{U} \cap \mathcal{G}_s \neq \emptyset}} \mathcal{X}(\mathcal{U}, \mathbf{c}) \leq 1; \\ & \text{For each RB } i, \quad \sum_{\substack{(\mathcal{U}, \mathbf{c}) \in \mathcal{M} \\ \mathbf{c}: i \in \mathbf{c}}} \mathcal{X}(\mathcal{U}, \mathbf{c}) \leq 1; \\ & \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{M}} \beta^q(\mathcal{U}, \mathbf{c}) \mathcal{X}(\mathcal{U}, \mathbf{c}) \leq 1, \quad 1 \leq q \leq J; \\ & \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{M}} \alpha^q(\mathcal{U}, \mathbf{c}) \mathcal{X}(\mathcal{U}, \mathbf{c}) \leq 1, \quad q \in \mathcal{I}, \end{aligned} \quad (3)$$

where  $\emptyset$  denotes the empty set and  $\mathcal{X}(\mathcal{U}, \mathbf{c})$  is an indicator function that returns one if users in  $\mathcal{U}$  are co-scheduled on the chunk indicated by  $\mathbf{c}$ . Note that the first constraint ensures that at-most one user is scheduled from each group and that each scheduled user is assigned at-most one chunk. In addition this constraint also enforces the complete overlap constraint. The second constraint enforces non-overlap among the assigned chunks. Note that  $p(\mathcal{U}, \mathbf{c})$  denotes the weighted sum-rate obtained upon co-scheduling the users in  $\mathcal{U}$  on the chunk indicated by  $\mathbf{c}$ . We emphasize that *there is complete freedom with respect to the computation of  $p(\mathcal{U}, \mathbf{c})$ . Indeed, it can accommodate finite buffer and practical MCS constraints, account for any particular receiver employed by the base station and can also incorporate any rule to assign a transmit antenna and a power level to each user in  $\mathcal{U}$  over the chunk  $\mathbf{c}$ .*

The first set of  $J$  knapsack constraints in (3), where  $J$  is arbitrary but fixed, are generic knapsack constraints. Without loss of generality, we assume that the weight of the pair  $(\mathcal{U}, \mathbf{c})$  in the  $q^{th}$  knapsack,  $\beta^q(\mathcal{U}, \mathbf{c})$ , lies in



the interval  $[0, 1]$ . Notice that we can simply drop each vacuous constraint, i.e., each constraint  $q$  for which  $\sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{M}} \beta^q(\mathcal{U}, \mathbf{c}) \leq 1$ . The second set of knapsack constraints are *column-sparse binary knapsack constraints*. In particular, for each pair  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}$  and  $q \in \mathcal{I}$  we have that  $\alpha^q(\mathcal{U}, \mathbf{c}) \in \{0, 1\}$ . Further, we have that for each  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}$ ,  $\sum_{q \in \mathcal{I}} \alpha^q(\mathcal{U}, \mathbf{c}) \leq \Delta$ , where  $\Delta$  is arbitrary but fixed and denotes the column-sparsity level. Note that here the cardinality of  $\mathcal{I}$  can scale polynomially in  $KN$  keeping  $\Delta$  fixed. Together these two sets of knapsack constraints can enforce a variety of practical constraints, including the control channel and the interference limit constraints. For instance, defining a generic knapsack constraint as  $\beta^1(\mathcal{U}, \mathbf{c}) = \frac{|\mathcal{U}|}{\tilde{K}}$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}$ , for any given input  $\tilde{K}$  can enforce that no more than  $\tilde{K}$  can be scheduled in a given interval, which represents a coarse control channel constraint. In a similar vein, consider any given choice of a victim adjacent base-station and a sub-band with the constraint that the total interference caused to the victim BS by users scheduled in the cell of interest, over all the RBs in the subband, should be no greater than a specified upper bound. This constraint can readily be modeled using a generic knapsack constraint where the weight of each pair  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}$  is simply the ratio of the total interference caused by users in  $\mathcal{U}$  to the victim BS over RBs that are in  $\mathbf{c}$  as well as the specified subband, and the specified upper bound. The interference is computed using the transmission parameters (such as the power levels, transmit antennas etc) that yield the metric  $p(\mathcal{U}, \mathbf{c})$ . A finer modeling of the LTE control channel constraints is more involved since it needs to employ the column-sparse knapsack constraints together with the user incompatibility constraints and is deferred to Appendix C.

Note that for a given  $K, N$ , an instance of the problem in (3) consists of a finite set  $\mathcal{I}$  of indices, a partition  $\{\mathcal{G}_1, \dots, \mathcal{G}_L\}$ , metrics  $\{p(\mathcal{U}, \mathbf{c})\} \forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}$  and weights  $\{\beta^q(\mathcal{U}, \mathbf{c})\}, \forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}, 1 \leq q \leq J$  and  $\{\alpha^q(\mathcal{U}, \mathbf{c})\}, \forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}, q \in \mathcal{I}$ . Then, in order to solve (3) for a given instance, we first partition the set  $\mathcal{M}$  into two parts as  $\mathcal{M} = \mathcal{M}^{\text{narrow}} \cup \mathcal{M}^{\text{wide}}$ , where we define  $\mathcal{M}^{\text{narrow}} = \{(\mathcal{U}, \mathbf{c}) \in \mathcal{M} : \beta^q(\mathcal{U}, \mathbf{c}) \leq 1/2, \forall 1 \leq q \leq J\}$  so that  $\mathcal{M}^{\text{wide}} = \mathcal{M} \setminus \mathcal{M}^{\text{narrow}}$ . We then define  $J$  sets,  $\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(J)}$  that cover  $\mathcal{M}^{\text{wide}}$  (note that any two of these sets can mutually overlap) as  $(\mathcal{U}, \mathbf{c}) \in \mathcal{V}^{(q)}$  iff  $\beta^q(\mathcal{U}, \mathbf{c}) > 1/2$  for  $q = 1, \dots, J$ . Recall that  $T, J$  are fixed and note that the cardinality of  $\mathcal{M}$ ,  $|\mathcal{M}|$ , is  $O(K^T N^2)$  and that  $\mathcal{M}^{\text{narrow}}$  and  $\{\mathcal{V}^{(q)}\}$  can be determined in polynomial time. Next, we propose Algorithm I which possesses the optimality given below. The complexity of Algorithm I, which is essentially determined by that of its module Algorithm IIa, scales polynomially in  $KN$  (recall that  $T$  is a constant) A detailed discussion on the complexity along with steps to reduce it are deferred to the next section. We offer the following theorem which is proved in Appendix A.

**Theorem 1.** *The problem in (3) is APX-hard, i.e., there is an  $\epsilon > 0$  such that it is NP hard to obtain a  $1 - \epsilon$  approximation algorithm for (3). Let  $\hat{W}^{\text{opt}}$  denote the optimal weighted sum rate obtained upon solving (3) and*



let  $\hat{W}$  denote the weighted sum rate obtained upon using Algorithm I. Then, we have that

$$\hat{W} \geq \begin{cases} \frac{\hat{W}^{\text{opt}}}{1+T+\Delta+2J}, & \text{If } \mathcal{M}^{\text{wide}} = \phi \\ \frac{\hat{W}^{\text{opt}}}{1+T+\Delta+3J}, & \text{Otherwise} \end{cases} \quad (4)$$

An interesting observation that follows from the proof of Theorem 1 is that any optimal allocation over  $\mathcal{M}^{\text{wide}}$  can include at-most one pair from each  $\mathcal{V}^{(q)}$ ,  $1 \leq q \leq J$ . Then since the number of pairs in each  $\mathcal{V}^{(q)}$ ,  $1 \leq q \leq J$  is  $O(K^T N^2)$ , we can determine an optimal allocation yielding  $\hat{W}^{\text{opt,wide}}$  via exhaustive enumeration with a high albeit polynomial complexity (recall that  $T$  and  $J$  are assumed to be fixed). Thus, by using exhaustive enumeration instead of Algorithm Iib, we can claim the following result.

**Corollary 1.** Let  $\hat{W}^{\text{opt}}$  denote the optimal weighted sum rate obtained upon solving (3) and let  $\hat{W}$  denote the weighted sum rate obtained upon using Algorithm II albeit with exhaustive enumeration over  $\mathcal{M}^{\text{wide}}$ . Then, we have that

$$\hat{W} \geq \begin{cases} \frac{\hat{W}^{\text{opt}}}{1+T+\Delta+2J}, & \text{If } \mathcal{M}^{\text{wide}} = \phi \\ \frac{\hat{W}^{\text{opt}}}{2+T+\Delta+2J}, & \text{Otherwise} \end{cases} \quad (5)$$

**Remark 1.** Some intuition on the process in the heart of Algorithm I (which is Algorithm IIa) is on order. Note that Algorithm IIa has two stages. The first one (comprising of steps 1 through 16) begins by initializing an empty stack  $\mathcal{S}$  and defining the current gain of each pair to be equal to its metric. Then, promising pairs are successively added to the top of the stack  $\mathcal{S}$ . Each time a pair is pushed into the stack, the current gain of each pair that can potentially be added and which conflicts with the pair just added (in terms of sharing a common RB or each having a user that belongs to an identical group or each having a unit weight in a common sparse knapsack constraint in  $\mathcal{I}$ ), is decremented by the current gain of the added pair. The idea behind this operation is that eventually only one pair among these conflicting pairs can be selected, so by decrementing the gains we ensure that a conflicting pair can be added in a later step only if it has a larger gain. Similarly, the gain of a non conflicting pair is also decremented by its maximal weight times twice the current gain of the added pair, in order account for the non-sparse knapsack constraints. At the end of the first stage the stack  $\mathcal{S}$  contains a set of promising pairs but the entire set need not be feasible for (3). In the second stage another stack  $\mathcal{S}'$  is formed by successively picking pairs from the top of stack  $\mathcal{S}$  and adding them to  $\mathcal{S}'$  if feasibility is satisfied. Note that the top down approach of picking pairs from  $\mathcal{S}$  is intuitively better since pairs at the top will have larger metrics than pairs below with whom they conflict.

For notational simplicity, henceforth unless otherwise mentioned, we assume that all users are mutually compatible, i.e.,  $L = K$  with  $\mathcal{G}_s = \{s\}$ ,  $1 \leq s \leq K$ .

### III. COMPLEXITY REDUCTION

In this section we present key techniques to significantly reduce the complexity of our proposed local ratio test based multi-user scheduling algorithm. As noted before the complexity of Algorithm I is dominated by that of its component Algorithm IIa. Accordingly, we focus our attention on Algorithm IIa and without loss of generality we assume that  $\mathcal{M} = \mathcal{M}^{\text{narrow}}$ . Notice that hitherto we have assumed that all the metrics  $\{p(\mathcal{U}, \mathbf{c}) : (\mathcal{U}, \mathbf{c}) \in \mathcal{M}\}$  are available. In practise, computing these  $O(K^T N^2)$  metrics, which are often complicated non-linear functions, is the main bottleneck and indeed must be accounted for in the complexity analysis. Before proceeding, we make the following assumption that is satisfied by all physically meaningful metrics.

**Assumption 1.** Sub-additivity: *We assume that for any  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}$*

$$p(\mathcal{U}, \mathbf{c}) \leq p(\mathcal{U}_1, \mathbf{c}) + p(\mathcal{U}_2, \mathbf{c}), \quad \forall \mathcal{U}_1, \mathcal{U}_2 : \mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2. \quad (6)$$

The following features can then be exploited for a significant reduction in complexity.

- *On demand metric computation:* Notice in Algorithm IIa that the metric for any  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}$ , where  $\text{Tail}(\mathbf{c}) = j$  for some  $j = 1, \dots, N$ , needs to be computed only at the  $j^{\text{th}}$  iteration at which point we need to determine

$$p'(\mathcal{U}, \mathbf{c}) = p(\mathcal{U}, \mathbf{c}) - \Gamma^{(j)}(\mathcal{U}, \mathbf{c}), \quad (7)$$

where the offset factor  $\Gamma^{(j)}(\mathcal{U}, \mathbf{c})$  is given by

$$\Gamma^{(j)}(\mathcal{U}, \mathbf{c}) = \sum_{(\mathcal{U}_m^*, \mathbf{c}_m^*) \in \mathcal{S}} \left( \tilde{p}(\mathcal{U}_m^*, \mathbf{c}_m^*) \mathcal{E}((\mathcal{U}, \mathbf{c}), (\mathcal{U}_m^*, \mathbf{c}_m^*)) + 2\tilde{p}(\mathcal{U}_m^*, \mathbf{c}_m^*) \max_{1 \leq q \leq J} \{\beta^q(\mathcal{U}_m^*, \mathbf{c}_m^*)\} \mathcal{E}^c((\mathcal{U}, \mathbf{c}), (\mathcal{U}_m^*, \mathbf{c}_m^*)) \right)$$

and where  $\tilde{p}(\mathcal{U}_m^*, \mathbf{c}_m^*)$  is equal to the  $p'(\mathcal{U}_m^*, \mathbf{c}_m^*)$  computed for the pair selected at the  $m^{\text{th}}$  iteration with  $m \leq j-1$  and  $\mathcal{E}((\mathcal{U}, \mathbf{c}), (\mathcal{U}_m^*, \mathbf{c}_m^*))$  denotes an indicator (with  $\mathcal{E}^c((\mathcal{U}, \mathbf{c}), (\mathcal{U}_m^*, \mathbf{c}_m^*)) = 1 - \mathcal{E}((\mathcal{U}, \mathbf{c}), (\mathcal{U}_m^*, \mathbf{c}_m^*))$ ) which is true when  $\mathcal{U}_m^* \cap \mathcal{U} \neq \emptyset$  or  $\mathbf{c} \cap \mathbf{c}_m^* \neq \emptyset$  or  $\exists q \in \mathcal{I} : \alpha^q(\mathcal{U}_m^*, \mathbf{c}_m^*) = \alpha^q(\mathcal{U}, \mathbf{c}) = 1$ . Further note that  $p'(\mathcal{U}, \mathbf{c})$  in (7) is required only if it is strictly positive. Then, an important observation is that if at the  $j^{\text{th}}$  iteration, we have already computed  $p(\mathcal{U}_1, \mathbf{c})$  and  $p(\mathcal{U}_2, \mathbf{c})$  for some  $\mathcal{U}_1, \mathcal{U}_2 : \mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ , then invoking the

sub-additivity property we have that

$$p'(\mathcal{U}, \mathbf{c}) \leq p(\mathcal{U}_1, \mathbf{c}) + p(\mathcal{U}_2, \mathbf{c}) - \Gamma^{(j)}(\mathcal{U}, \mathbf{c}), \quad (8)$$

so that if the RHS in (8) is not strictly positive or if it is less than the greatest value of  $p'(\mathcal{U}', \mathbf{c}')$  computed in the current iteration for some other pair  $(\mathcal{U}', \mathbf{c}') : \text{Tail}(\mathbf{c}') = j$ , then *we do not need to compute  $p'(\mathcal{U}, \mathbf{c})$  and hence the metric  $p(\mathcal{U}, \mathbf{c})$ .*

- *Selective update* Note that in the  $j^{\text{th}}$  iteration, once the best pair  $(\mathcal{U}_j^*, \mathbf{c}_j^*)$  is selected and it is determined that  $p'(\mathcal{U}_j^*, \mathbf{c}_j^*) > 0$ , we need to update the metrics for pairs  $(\mathcal{U}', \mathbf{c}') : \text{Tail}(\mathbf{c}') \geq j + 1$ , since only such pairs will be considered in future iterations. Thus, the offset factors  $\{\Gamma^{(j)}(\mathcal{U}', \mathbf{c}')\}$  need to be updated only for such pairs, via

$$\Gamma^{(j+1)}(\mathcal{U}', \mathbf{c}') = \Gamma^{(j)}(\mathcal{U}', \mathbf{c}') + p'(\mathcal{U}_j^*, \mathbf{c}_j^*) \mathcal{E}((\mathcal{U}', \mathbf{c}'), (\mathcal{U}_j^*, \mathbf{c}_j^*)) + 2p'(\mathcal{U}_j^*, \mathbf{c}_j^*) \max_{1 \leq q \leq J} \{\beta^q(\mathcal{U}', \mathbf{c}')\} \mathcal{E}^c((\mathcal{U}', \mathbf{c}'), p'(\mathcal{U}_j^*, \mathbf{c}_j^*)).$$

Further, if by exploiting sub-additivity we can deduce that  $p'(\mathcal{U}', \mathbf{c}') \leq 0$  for any such pair, then we can drop such a pair along with its offset factor from future consideration.

#### IV. IMPROVING PERFORMANCE VIA A SECOND PHASE

A potential drawback of the LRT based algorithm is that some RBs may remain un-utilized, i.e., they may not be assigned to any user. Notice that when the final stack  $\mathcal{S}'$  is built in the while-loop of Algorithm IIa, an allocation or pair from the top of stack  $\mathcal{S}$  is added to stack  $\mathcal{S}'$  only if it does not violate feasibility when considered together with those already in stack  $\mathcal{S}'$ . Often multiple pairs from  $\mathcal{S}$  are dropped due to such feasibility violations, resulting in spectral holes formed by unassigned RBs. To mitigate this problem, we perform a second phase. The second phase consists of running Algorithm IIa again albeit with modified metrics  $\{\check{p}(\mathcal{U}, \mathbf{c}) : (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}\}$  which are obtained via the following steps.

- 1) Initialize  $\check{p}(\mathcal{U}, \mathbf{c}) = p(\mathcal{U}, \mathbf{c})$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$ . Let  $\mathcal{S}'$  be obtained as the output of Algorithm IIa when it is implemented first.
- 2) For each  $(\mathcal{U}, \mathbf{c}) \in \mathcal{S}'$ , we ensure that any user in  $\mathcal{U}$  is not scheduled by phase two in any other user set save  $\mathcal{U}$ , by setting

$$\check{p}(\mathcal{U}', \mathbf{c}') = 0 \text{ if } \mathcal{U}' \neq \mathcal{U} \text{ \& } \mathcal{U}' \cap \mathcal{U} \neq \phi, \forall (\mathcal{U}', \mathbf{c}') \in \mathcal{M}^{\text{narrow}}. \quad (9)$$

- 3) For each  $(\mathcal{U}, \mathbf{c}) \in \mathcal{S}'$ , we ensure that no other user set save  $\mathcal{U}$  is assigned any RB in  $\mathbf{c}$ , by setting

$$\check{p}(\mathcal{U}', \mathbf{c}') = 0 \text{ if } \mathcal{U}' \neq \mathcal{U} \text{ \& } \mathbf{c}' \cap \mathbf{c} \neq \emptyset, \forall (\mathcal{U}', \mathbf{c}') \in \mathcal{M}^{\text{narrow}}. \quad (10)$$

- 4) For each  $(\mathcal{U}, \mathbf{c}) \in \mathcal{S}'$ , we ensure that the allocation  $(\mathcal{U}, \mathbf{c})$  is either unchanged by phase two or is expanded, by setting

$$\check{p}(\mathcal{U}, \mathbf{c}') = \begin{cases} p(\mathcal{U}, \mathbf{c}'), & \text{If Tail}(\mathbf{c}') \geq \text{Tail}(\mathbf{c}) \text{ \& } \text{Head}(\mathbf{c}') \leq \text{Head}(\mathbf{c}) \\ 0, & \text{Otherwise} \end{cases}$$

A consequence of using the modified metrics is that the second phase has a significantly less complexity since a large fraction of the allocations are disallowed (since many of the modified metrics are zero). While the second phase does not offer any improvement in the approximation factor, simulation results presented in the sequel reveal that it offers a good performance improvement with very low complexity addition.

## V. SIMULATION RESULTS: SINGLE CELL SETUP

In this section we evaluate key features of our proposed algorithm over an idealized single-cell setup. In particular, we simulate an uplink wherein the BS is equipped with four receive antennas. The system has 280 sub-carriers divided into 20 RBs (of size 14 sub-carriers each) available as data subcarriers that are used for serving the users. We assume 10 active users all of whom have identical maximum transmit powers. We model the fading channel between each user and the BS as a six-path equal gain i.i.d. Rayleigh fading channel. In all the results given below we assume an infinitely backlogged traffic model. For simplicity, we assume that there are no knapsack constraints and that at-most two users can be co-scheduled on an RB (i.e.,  $J = 0, \Delta = 0$  and  $T = 2$ ). Further, each user can employ ideal Gaussian codes and upon being scheduled, divides its maximum transmit power equally among its assigned RBs. Notice that since  $\mathcal{M} = \mathcal{M}^{\text{narrow}}$  we can directly use Algorithm IIa.

In Fig. 2, we plot the average cell spectral efficiency (in bits-per-sec-per-Hz) versus the average transmit SNR (dB) for an uplink where each user has one transmit antenna and the BS employs the linear MMSE receiver. We plot the spectral efficiencies achieved when Algorithm IIa is employed with and without the second phase (described in Section IV), respectively (denoted in the legend by MU-MMSE-LRT-2Step and MU-MMSE-LRT-1Step). Also plotted is the upper bound obtained by the linear programming (LP) relaxation of (3) along with the spectral efficiency obtained upon rounding the LP solution to ensure feasibility (denoted in the legend by

MU-MMSE-LP-UB and MU-MMSE-LP-Rounding, respectively). In Fig. 3, we plot the average cell spectral efficiency versus the average transmit SNR for an uplink where each user has one transmit antenna and the BS employs the successive interference cancellation (SIC) receiver. We plot the spectral efficiencies achieved when Algorithm IIa is employed with and without the second phase, respectively (denoted in the legend by MU-SIC-LRT-2Step and MU-SIC-LRT-1Step). Also plotted are the corresponding LP upper bound along with the spectral efficiency obtained upon rounding the LP solution. Figure 4 and Figure 5 are the counterparts of Figure 2 and Figure 3, respectively, but where each user has two transmit antennas and the BS can thus exploit transmit antenna selection. Finally, in Fig. 6 we plot the normalized spectral efficiencies obtained by dividing each spectral efficiency by the one yielded by Algorithm IIa when only single user (SU) scheduling is allowed, which in turn can be emulated by setting all metrics  $p(\mathcal{U}, \mathbf{c}) : (\mathcal{U}, \mathbf{c}) \in \mathcal{M}$  in (3) to be zero whenever  $|\mathcal{U}| \geq 2$ .<sup>1</sup> In all considered schemes we assume that Algorithm IIa with the second phase is employed. From Figures 2 to 6, we have the following observations:

- For both SIC and MMSE receivers, the performance of Algorithm IIa is more than 80% of the respective LP upper bounds, which is much superior to the worst case guarantee  $1/3$  (obtained by specializing the result in (4) by setting  $\mathcal{M}^{\text{wide}} = \phi, T = 2$  and  $\Delta = J = 0$ ). Further, for both the receivers the performance of Algorithm IIa with the second phase is more than 90% the respective LP upper bounds. The same conclusions can be drawn when antenna selection is also exploited by the BS. In all cases, the performance of LP plus rounding scheme is exceptional and within 2% of the respective upper bound. However the complexity of this LP seems unaffordable as yet for practical implementation.<sup>2</sup>
- The SIC receiver results in a small gain (1.5% to 2.5%) over the MMSE receiver. This gain will increase if we consider more correlated fading over which the limitation of linear receivers is exposed and as the maximum number of users that can be co-scheduled on an RB ( $T$ ) is increased since the SIC allows for improved system rates via co-scheduling a larger number of users on an RB, whereas the MMSE will become interference limited. Note that antenna selection seems to provide a much larger gain (6% to 8%) than the one offered by the advanced SIC receiver. This observation must be tempered by the facts that the simulated scenario of independent (uncorrelated) fading is favorable for antenna selection and that the antenna switching loss (about 0.5 dB in practical devices) as well as the additional pilot overhead have been neglected.

<sup>1</sup>Note that for SU scheduling MMSE and SIC receivers are equivalent.

<sup>2</sup>For instance, this LP involves about 11,500 variables and must be solved within each scheduling interval whose duration in LTE systems is one millisecond.

- MU scheduling offers substantial gains over SU scheduling (ranging from 50% to 75% for the considered SNRs). This follows since the degrees of freedom available here for MU scheduling is twice that of SU-scheduling.

Next, in Fig. 7 we plot the normalized complexities for the scheduling schemes considered in Figures 2 to 5. Here the complexity of a scheduling scheme is determined by the complexity of the metric computations made by it. In all cases the second phase is performed for Algorithm IIa and more importantly the sub-additivity property together with the on-demand metric computation feature are exploited, as described in Section III, to avoid redundant metric computations. All schemes compute the metrics  $\{p(\mathcal{U}, \mathbf{c}) : (\mathcal{U}, \mathbf{c}) \in \mathcal{M} \text{ \& } |\mathcal{U}| = 1\}$  and each such metric is deemed to have unit complexity when each user has one transmit antenna and a complexity of two units when each user two transmit antennas. On the other hand, for each evaluated  $p(\mathcal{U}, \mathbf{c}) : |\mathcal{U}| = 2$ , the complexity is taken to be two units when each user has one transmit antenna and the BS employs the MMSE receiver and one unit when each user has one transmit antenna and the BS employs the SIC receiver. The latter stems from the fact that with the SIC receiver, one of the users sees an interference free channel. Thus, its contribution to the metric is equal to the already computed single-user metric determined for the allocation when that user is scheduled alone on the corresponding chunk. Similarly, for each evaluated  $p(\mathcal{U}, \mathbf{c}) : |\mathcal{U}| = 2$ , the complexity is taken to be eight units when each user has two transmit antennas and the BS employs transmit antenna selection together with the MMSE receiver and four units for the case when the BS employs transmit antenna selection together with the SIC receiver. Note that MMSE-Total, SIC-Total denote the complexities obtained by counting the corresponding complexities for all pairs  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}$ , respectively, whereas MMSE-AS-Total, SIC-AS-Total denote the total complexities obtained when antenna selection is employed by the BS together with the MMSE receiver and the SIC receiver, respectively. Note that all complexities in Fig. 7 are normalized by MMSE-AS-Total. The key takeaway from Fig. 7 is that exploiting sub-additivity together with the on-demand metric computation can result in very significant complexity reduction. In particular, as per our definition of complexity, more than 80% reduction can be obtained for the MMSE receiver and more than 75% reduction can be obtained for the SIC receiver, with the respective gains being larger when antenna selection is also exploited. Further, we note that considering Algorithm IIa, the second phase itself adds a very small complexity overhead but results in a large performance improvement. To illustrate this, for the MMSE receiver the complexity overhead ranges from 2 to 4%, whereas the performance improvement ranges from 9 to 13%, respectively. Finally, in Fig. 8 we conduct a complexity comparison identical to that in Fig. 7 except that the complexity computed for each  $p(\mathcal{U}, \mathbf{c})$  is now also multiplied by the size of the chunk indicated by  $\mathbf{c}$ . Notice

that the complexity reductions achieved by exploiting sub-additivity property together with the on-demand metric computation feature are now even larger.

## VI. SEQUENTIAL LRT BASED MU SCHEDULING

We next propose a sequential LRT based MU scheduling method that yields a scheduling decision over  $\mathcal{M}^{\text{narrow}}$ . As before, our focus is on avoiding as many metric computations as possible. The idea is to implement the LRT based MU scheduling algorithm in  $T$  iterations, where we recall  $T$  denotes the maximum number of users that can be co-scheduled on an RB. In particular, in the first iteration we define metrics  $\check{p}(\mathcal{U}, \mathbf{c}) = p(\mathcal{U}, \mathbf{c})$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}} : |\mathcal{U}| = 1$  with  $\check{p}(\mathcal{U}, \mathbf{c}) = 0$  otherwise, and use these metrics in Algorithm IIa to obtain a tentative scheduling decision. Further, in the  $s^{\text{th}}$  iteration where  $2 \leq s \leq T - 1$ , we first perform the following steps to obtain metrics  $\check{p}(\mathcal{U}, \mathbf{c})$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$ , where only a few of these metrics are positive, and then use them in Algorithm IIa to obtain a tentative decision.

- Initialize  $\check{p}(\mathcal{U}, \mathbf{c}) = 0$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$ . Let  $\mathcal{S}'$  denote the output obtained from the previous iteration.
- For each  $(\mathcal{U}, \mathbf{c}) \in \mathcal{S}'$  we ensure that any user in set  $\mathcal{U}$  can be scheduled in the  $s^{\text{th}}$  iteration only as part of a set that contains all users in  $\mathcal{U}$  along with at-most one additional user, by setting

$$\check{p}(\mathcal{U}', \mathbf{c}') = 0 \text{ if } (\mathcal{U} \not\subseteq \mathcal{U}' \text{ \& } \mathcal{U}' \cap \mathcal{U} \neq \phi) \text{ or } (|\mathcal{U}'| > |\mathcal{U}| + 1 \text{ \& } \mathcal{U}' \cap \mathcal{U} \neq \phi), \quad \forall (\mathcal{U}', \mathbf{c}') \in \mathcal{M}^{\text{narrow}}. \quad (11)$$

- For each  $(\mathcal{U}, \mathbf{c}) \in \mathcal{S}'$ , we also ensure that any user in set  $\mathcal{U}$  must be assigned all RBs in  $\mathbf{c}$ , by considering each  $(\mathcal{U}', \mathbf{c}') \in \mathcal{M}^{\text{narrow}} : \mathcal{U} \subseteq \mathcal{U}' \text{ \& } |\mathcal{U}'| \leq |\mathcal{U}| + 1$ , and setting

$$\check{p}(\mathcal{U}', \mathbf{c}') = \begin{cases} p(\mathcal{U}', \mathbf{c}'), & \text{If Tail}(\mathbf{c}') \geq \text{Tail}(\mathbf{c}) \text{ \& } \text{Head}(\mathbf{c}') \leq \text{Head}(\mathbf{c}) \\ 0, & \text{Otherwise} \end{cases} \quad (12)$$

$$(13)$$

In the last iteration, i.e. when  $s = T$ , we initialize  $\check{p}(\mathcal{U}, \mathbf{c}) = p(\mathcal{U}, \mathbf{c})$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$ . Then, using the set  $\mathcal{S}'$  obtained as the output of the  $(T - 1)^{\text{th}}$  iteration, we perform the two aforementioned steps. Additionally, to ensure non-overlapping chunk allocation, for each  $(\mathcal{U}, \mathbf{c}) \in \mathcal{S}'$  we set

$$\check{p}(\mathcal{U}', \mathbf{c}') = 0 \text{ if } \mathbf{c}' \cap \mathbf{c} \neq \phi \text{ \& } \mathcal{U}' \cap \mathcal{U} = \phi, \quad \forall (\mathcal{U}', \mathbf{c}') \in \mathcal{M}^{\text{narrow}}. \quad (14)$$

Note that the different initialization chosen for the last iteration seeks to select a larger pool of positive metrics and can improve performance albeit at an increased complexity. In addition, after each iteration we also enforce an *improvement condition* which checks if the weighted sum rate yielded by the obtained decision is strictly



greater than that computed at the end of the previous iteration. If this condition is satisfied, we proceed to the next iteration, else the process is terminated and the solution obtained at the end of the previous iteration is returned. Notice that in each iteration only a small subset out of the set of all metrics is selected, which in particular is that whose corresponding pairs are compatible (as defined in the aforementioned conditions) with the output tentative scheduling decision of the previous iteration. Next, we offer an approximation result for the sequential LRT based MU scheduling that holds under mild assumptions.

**Assumption 2.** Suppose  $\mathcal{F}$  is any allocation  $\{(\mathcal{U}, \mathbf{c})\}$  that is feasible for (3). Then  $\mathcal{F}$  is downward closed in the following sense. Any allocation  $\mathcal{F}'$  constructed as  $\mathcal{F}' = \{(\mathcal{U}', \mathbf{c}) : \mathcal{U}' \subseteq \mathcal{U} \text{ \& } (\mathcal{U}, \mathbf{c}) \in \mathcal{F}\}$  is also feasible.

**Proposition 1.** Suppose that Assumptions 1 and 2 are satisfied. Let the weighted sum rate yielded by the sequential LRT based MU scheduling over  $\mathcal{M}^{\text{narrow}}$  be denoted by  $\hat{W}^{\text{seq-narrow}}$ . Then,

$$\hat{W}^{\text{seq-narrow}} \geq \frac{\hat{W}^{\text{opt,narrow}}}{T(2 + \Delta + 2J)}. \quad (15)$$

*Proof:* Let  $\mathcal{F}^{\text{opt,narrow}}$  be an optimal allocation of pairs from  $\mathcal{M}^{\text{narrow}}$  that yields a weighted sum rate  $\hat{W}^{\text{opt,narrow}}$  and initialize  $\mathcal{F}' = \phi$ . Then for each  $(\mathcal{U}, \mathbf{c}) \in \mathcal{F}^{\text{opt,narrow}}$  determine the best user  $\hat{u} = \arg \max_{u \in \mathcal{U}} \{p(u, \mathbf{c})\}$  and insert the pair  $(\hat{u}, \mathbf{c})$  into  $\mathcal{F}'$ . Note that due the sub-additivity property in Assumption 1, we must have that  $p(\hat{u}, \mathbf{c}) \geq \frac{p(\mathcal{U}, \mathbf{c})}{T}$ . Consequently, we have that the weighted sum rate yielded by  $\mathcal{F}'$  is at-least  $\frac{\hat{W}^{\text{opt,narrow}}}{T}$ . Furthermore, on account of Assumption 2,  $\mathcal{F}'$  is a feasible allocation for (3). Then, suppose  $\mathcal{F}^{(1)}$  is the allocation obtained after the first iteration of the sequential algorithm. Since this allocation is a result of applying Algorithm IIa with single user metrics, upon invoking Theorem 1 we can claim that the weighted sum rate yielded by  $\mathcal{F}^{(1)}$  is at-least a fraction  $\frac{1}{2+\Delta+2J}$  of the best single-user allocation, where a single-user allocation is one where each pair includes only one user. Then, since  $\mathcal{F}'$  is one such single-user allocation we can claim that the weighted sum rate yielded by  $\mathcal{F}^{(1)}$  is at-least  $\frac{\hat{W}^{\text{opt,narrow}}}{T}$ . Finally, since the improvement condition ensures that the weighted sum rates yielded by tentative allocations across iterations are monotonically increasing, we can deduce that the proposition is true.

## VII. USER PRE-SELECTION

In a practical cellular system the number of active users can be large. Indeed the control channel constraints may limit the BS to serve a much smaller subset of users. It thus makes sense from a complexity stand-point to pre-select a pool of *good* users and then use the MU scheduling algorithm on the selected pool of users. Here

we propose a few user pre-selection algorithms. For convenience, wherever needed, we assume that at-most two users can be co-scheduled on an RB (i.e.,  $T = 2$ ) which happens to be the most typical value.

Before proceeding we need to define some terms that will be required later. Suppose that each user has one transmit antenna and let  $\mathbf{h}_{u,j}$  denote the effective channel vector seen at the BS from user  $u$  on RB  $j$ , where  $1 \leq u \leq K$  and  $1 \leq j \leq N$ . Note that the effective channel vector includes the fading as well as the path loss factor and a transmit power value. Then, letting  $w_u$  denote the PF weight of user  $u$ , we define the following metrics:

- Consider first the weighted rate that the system can obtain when it schedules user  $u$  alone on RB  $j$ ,

$$p^{\text{su}}(u, j) = w_u \log(1 + \mathbf{h}_{u,j}^\dagger \mathbf{h}_{u,j}), \quad \forall 1 \leq u \leq K \text{ \& } 1 \leq j \leq N. \quad (16)$$

- Let  $\mathcal{U} = \{u, v\} : u \neq v$  be any pair of users and suppose that the BS employs the MMSE receiver. Then, the weighted sum rate obtained by scheduling the user pair  $\mathcal{U}$  on RB  $j$  is given by

$$p^{\text{mmse}}(\mathcal{U}, j) = w_u \log(1 + \mathbf{h}_{u,j}^\dagger (\mathbf{I} + \mathbf{h}_{v,j} \mathbf{h}_{v,j}^\dagger)^{-1} \mathbf{h}_{u,j}) + w_v \log(1 + \mathbf{h}_{v,j}^\dagger (\mathbf{I} + \mathbf{h}_{u,j} \mathbf{h}_{u,j}^\dagger)^{-1} \mathbf{h}_{v,j}). \quad (17)$$

- Finally, assume that the BS employs the SIC receiver and let  $\hat{u} = \arg \max_{s \in \mathcal{U}} \{w_s\}$  and let  $\hat{v} = \mathcal{U} \setminus \hat{u}$ . Then, the weighted sum rate obtained by scheduling the user pair  $\mathcal{U}$  on RB  $j$  is given by

$$p^{\text{sic}}(\mathcal{U}, j) = p^{\text{su}}(\hat{u}, j) + w_{\hat{v}} \log(1 + \mathbf{h}_{\hat{v},j}^\dagger (\mathbf{I} + \mathbf{h}_{\hat{u},j} \mathbf{h}_{\hat{u},j}^\dagger)^{-1} \mathbf{h}_{\hat{v},j}). \quad (18)$$

We are now ready to offer our user pre-selection rules where a pool of  $\tilde{K}$  users must be selected from the  $K$  active users. Notice that to reduce complexity, all rules neglect the contiguity and the complete overlap constraints.

- 1) The first rule simply selects the  $\tilde{K}$  users that offer the  $\tilde{K}$  largest single-user rates among  $\{\sum_{j=1}^N p^{\text{su}}(u, j)\}_{u=1}^K$ .
- 2) The second rule assumes that each RB can be assigned to at-most one user. Then, if a user subset  $\mathcal{A} \subseteq \{1, \dots, K\}$  is selected, the system weighted sum-rate is given by

$$f(\mathcal{A}) \triangleq \sum_{j=1}^N \max_{u \in \mathcal{A}} \{p^{\text{su}}(u, j)\}. \quad (19)$$

It can be shown that  $f : 2^{\{1, \dots, K\}} \rightarrow \mathbb{R}_+$  is a *monotonic sub-modular set function* [16]. As a result, the user pre-selection problem

$$\arg \max_{\mathcal{A} \subseteq \{1, \dots, K\} : |\mathcal{A}| \leq \tilde{K}} \{f(\mathcal{A})\} \quad (20)$$

can be sub-optimally solved by adapting a simple greedy algorithm [24], which offers a half approximation [16].

- 3) The third rule assumes that each RB can be assigned to at-most two users and that the BS employs the MMSE receiver. Then, if a user subset  $\mathcal{A} \subseteq \{1, \dots, K\}$  is selected, the system weighted sum-rate is given by

$$g(\mathcal{A}) \triangleq \sum_{j=1}^N \max \left\{ \max_{u \in \mathcal{A}} \{p^{\text{su}}(u, j)\}, \max_{\substack{\mathcal{U}=(u,v): u < v \\ u, v \in \mathcal{A}}} \{p^{\text{mmse}}(\mathcal{U}, j)\} \right\}. \quad (21)$$

It can be shown that  $g : 2^{\{1, \dots, K\}} \rightarrow \mathbb{R}_+$  is a monotonic set function but unfortunately it need not be sub-modular. Nevertheless, we proceed to employ the greedy algorithm to sub-optimally solve

$$\arg \max_{\mathcal{A} \subseteq \{1, \dots, K\}: |\mathcal{A}| \leq \tilde{K}} \{g(\mathcal{A})\} \quad (22)$$

- 4) The fourth rule also assumes that each RB can be assigned to at-most two users but that the BS employs the SIC receiver. However, even upon replacing  $p^{\text{mmse}}(\mathcal{U}, j)$  in (21) with  $p^{\text{sic}}(\mathcal{U}, j)$ , the resulting set function need not be sub-modular. As a result we use a different metric. In particular, for a user subset  $\mathcal{A} \subseteq \{1, \dots, K\}$  we employ a metric that is given by

$$\begin{aligned} h(\mathcal{A}) &\triangleq \sum_{j=1}^N \sum_{\substack{\mathcal{U}=(u,v): u \leq v \\ u, v \in \{1, \dots, K\}}} (p^{\text{su}}(\mathcal{U} \cap \mathcal{A}, j) \mathcal{X}(|\mathcal{U} \cap \mathcal{A}| = 1) + p^{\text{sic}}(\mathcal{U}, j) \mathcal{X}(|\mathcal{U} \cap \mathcal{A}| = 2)) \\ &= \sum_{j=1}^N \left( (K - |\mathcal{A}| + 1) \sum_{u \in \mathcal{A}} p^{\text{su}}(u, j) + \sum_{\substack{\mathcal{U}=(u,v): u < v \\ u, v \in \mathcal{A}}} p^{\text{sic}}(\mathcal{U}, j) \right). \end{aligned} \quad (23)$$

Notice that for any  $\mathcal{A}$ ,  $h(\mathcal{A})$  represents the system weighted sum-rate when time-sharing is employed by the system wherein in each slot only a particular user or two distinct users from a particular pair in  $\{1, \dots, K\}$  are allowed to be scheduled. Then, a key result which is proved in Appendix B, is the following.

**Proposition 2.** *The set function  $h(\cdot)$  defined in (23) is a monotonic sub-modular set function. Thus the problem*

$$\arg \max_{\mathcal{A} \subseteq \{1, \dots, K\}: |\mathcal{A}| \leq \tilde{K}} \{h(\mathcal{A})\} \quad (24)$$

*can be solved sub-optimally (with a 1/2 approximation) by a simple greedy algorithm.*

As a benchmark to compare the performance of the proposed user pre-selection algorithms we can consider the case where LRT MU scheduling is employed without user pre-selection but where an additional knapsack

constraint is used to enforce the limit on the number of users that can be scheduled in an interval. It can be verified that this can be achieved by defining a knapsack constraint in (3) as  $\beta^1(\mathcal{U}, \mathbf{c}) = \frac{|\mathcal{U}|}{K}$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}$ .

### VIII. SYSTEM LEVEL SIMULATION RESULTS

We now present the performance of our MU scheduling algorithms via detailed system level simulations. The simulation parameters conform to those used in 3GPP LTE evaluations and are given in Table IV. In all cases inter-cell interference suppression (IRC) is employed by each base-station (BS).

We first consider the case when each cell (or sector) has an average of 10 users and where there are no knapsack constraints. In Table V we report the cell average and cell edge spectral efficiencies. The percentage gains shown for the MU scheduling schemes are over the baseline LRT based single-user scheduling scheme. Note that for the first three scheduling schemes we employed the second phase described in Section IV. Also, we observed that the LRT based SU scheduling together with the second phase yields at-least as good a performance (for both cell-edge and cell average throughputs) as those of the deterministic SU scheduling algorithms in [17], [18], so we have omitted results for the latter algorithms. As seen from Table V, MU scheduling in conjunction with an advanced SIC receiver at the BS can result in very significant gains in terms of cell average throughput (about 27%) along with good cell edge gains. For the simpler MMSE receiver, we see significant cell average throughput gains (about 18%) but a degraded cell edge performance. We note that it is possible to tradeoff a small fraction of the cell edge gains for a large cell edge performance improvement by altering the PF rule. Finally, the last two reported schemes are based on the sequential-LRT method described in Section VI. We notice that sequential-LRT based scheduling provides significant cell average gains while retaining the cell edge performance of SU scheduling. Thus, the sequential LRT based scheduling method is an attractive way to tradeoff some cell average throughput gains for a reduction in complexity.

Next, in Tables VI and VII we consider LRT based MU scheduling, with the second phase described in Section IV, for the case when the BS employs the MMSE receiver and the case when it employs the SIC receiver, respectively. In each case we assume that an average of 15 users are present in each cell and at-most 7 first-transmission users can be scheduled in each interval. Thus, a limit on the number of scheduled users might have to be enforced in each scheduling interval. As a benchmark, we enforce this constraint (if it is required) using one knapsack constraint as described in Section VII. Note that upon specializing the result in Theorem 1 (with  $\mathcal{M}^{\text{wide}} = \phi$ ,  $T = 2$  and  $\Delta = 0$ ,  $J = 1$ ) we see that the LRT based MU scheduling algorithm guarantees an approximation factor of  $1/5$ . Then, we examine the scenario where a pool of  $\tilde{K} = 7$  users is pre-selected whenever the number of first-transmission users is larger than 7. The LRT based MU scheduling algorithm is then

employed on this pool without any constraints. In Table VI we have used the first second and third pre-selection rules from Section VII whereas in Table VII we have used the first second and fourth pre-selection rules. It is seen that the simple rule one provides a superior performance compared to the benchmark. Indeed, it is attractive since it involves computation of only single user metrics. The other rule (rule 2) which possess this feature, however provides much less improvement mainly because it is much more aligned to single user scheduling. Rules 3 and 4 involve computation of metrics that involve user-pairing and hence incur higher complexity. For the MMSE receiver, the gain of rule 3 over rule 1 is marginal mainly because the metric in rule 3 is not sub-modular and hence cannot be well optimized by the simple greedy rule. On the other hand, considering the MMSE receiver, the gain of rule 4 over rule 1 is larger because the metric used in rule 4 is indeed sub-modular and hence can be well optimized by the simple greedy rule.

## IX. CONCLUSIONS

We considered resource allocation in the 3GPP LTE cellular uplink which allows for transmit antenna selection for each scheduled user as well as multi-user scheduling, wherein multiple users can be assigned the same time-frequency resource. We showed that the resulting resource allocation problem, which must comply with several practical constraints, is NP-hard. We then proposed constant-factor polynomial-time approximation algorithms and demonstrated their performance via simulations.

## REFERENCES

- [1] 3GPP, "TSG-RAN EUTRA, rel.8," *TR 36.101*, Jun 2011.
- [2] W. Yu and W. Rhee, "Degrees of freedom in wireless multiuser spatial multiplex systems with multiple antennas," *IEEE Trans. Commun.*, vol. 54, pp. 1747–1753, Oct 2006.
- [3] D. Tse and S. Hanly, "Multiaccess fading channels-part I: Polymatroid structure, optimal resource allocation, and throughput capacities," *IEEE Trans. Inform. Theory*, 1998.
- [4] W. Yu and J. Cioffi, "Constant power water-filling: Performance bound and low-complexity implementation," *IEEE Trans. Commun.*, vol. 54, pp. 23–28, Jan. 2006.
- [5] A. Barnoy, R. Bar-Yehuda, A. Freund, J. Naor, and B. Schieber, "A unified approach to approximating resource allocation and scheduling," *ACM Symposium on Theory of Computing*, 2000.
- [6] H. Yang, F. Ren, C. Lin, and J. Zhang, "Frequency-domain packet scheduling for 3GPP LTE uplink," *IEEE Infocom*, 2010.
- [7] W. Yu and R. Liu, "Dual methods for nonconvex spectrum optimization of multicarrier systems," *IEEE Trans. Commun.*, vol. 54, pp. 1310–1322, July 2006.
- [8] W. Yu, R. Liu, and R. Cendrillon, "Dual optimization methods for multiuser orthogonal frequency division multiplex systems," *Proc. IEEE Global Telecommun. Conf. (Globecom)*, 2004.

- [9] C. Wong, R. Cheng, K. Letaief, and R. Murch, "Multiuser OFDM with adaptive subcarrier, bit, and power allocation," *IEEE J. Select. Areas Commun.*, vol. 17, pp. 1747 – 1758, Oct. 1999.
- [10] S. Lee, S. Choudhury, A. Khoshnevis, S. Xu, and S. Lu, "Downlink MIMO with frequency-domain packet scheduling for 3GPP LTE," *IEEE Infocom*, 2009.
- [11] A. Abrardo, P. Detti, and M. Moretti, "Message passing resource allocation for the uplink of multicarrier systems," *Proc. IEEE Int. Conf. on Commun. (ICC)*, 2009.
- [12] A. Lozano, A. M. Tulino, and S. Verdu, "Optimum power allocation for parallel Gaussian channels with arbitrary input distributions," *IEEE Trans. Inform. Theory*, vol. 52, pp. 3303–3351, July 2006.
- [13] W. Noh, "A distributed resource control for fairness in ofdma systems: English-auction game with imperfect information," *Proc. IEEE Global Telecommun. Conf. (Globecom)*, 2008.
- [14] H. Zhu, Z. Li, and K. J. R. Liu, "Fair multiuser channel allocation for OFDMA networks using Nash bargaining solutions and coalitions," *IEEE Trans. Commun.*, vol. 53, pp. 1336–1376, Aug. 2005.
- [15] K. Kim and Y. Han, "Joint subcarrier and power allocation in uplink OFDMA systems," *IEEE Commun. Let.*, vol. 9, pp. 526–528, Jan. 2006.
- [16] H. Zhang, N. Prasad, and S. Rangarajan, "MIMO downlink scheduling in LTE and LTE-advanced systems," in *Proc. 2012 IEEE INFOCOM Miniconference*, (Orlando, FL), Mar. 2012.
- [17] M. Andrews and L. Zhang, "Multiserver scheduling with contiguity constraints," *Proc. IEEE Infocom*, 2009.
- [18] S. B. Lee, I. Pefkianakis, A. Meyerson, X. Shugong, and L. Songwu, "Proportional fair frequency-domain packet scheduling for 3GPP LTE uplink," in *Proc. IEEE INFOCOM*, 2009.
- [19] N. Prasad, H. Zhang, M. Jiang, G. Yue, and S. Rangarajan, "Resource allocation in 4G MIMO cellular uplink," *IEEE Globecom*, 2011.
- [20] R. Madan and S. Ray, "Uplink resource allocation for frequency selective channels and fractional power control," in *Proc. IEEE ICC*, (Kyoto, Japan), 2011.
- [21] K. Yang, N. Prasad, and X. Wang, "A message-passing approach to distributed resource allocation in uplink DFT-spread-OFDMA systems," *IEEE Trans. Commun.*, vol. 59, pp. 1099–1113, Apr. 2011.
- [22] J. Huang, V. G. Subramanian, R. Agrawal, and R. Berry, "Joint scheduling and resource allocation in uplink OFDM systems," *IEEE J. Select. Areas Commun.*, vol. 27, pp. 226–234, Feb. 2009.
- [23] Y. Liu and E. Knightly, "Opportunistic fair scheduling over multiple wireless channels," in *Proc. 2003 IEEE INFOCOM*, (San Francisco, CA), Mar. 2003.
- [24] G. L. Nemhauser and L. A. Wolsey, "Best algorithms for approximating the maximum of a submodular set function," *Math. Operations Research*, 1978.

## APPENDIX A

### PROOF OF THEOREM 1

Let us specialize (3) to instances where all the knapsack constraints are vacuous, where  $L = K$  and  $\mathcal{G}_s = \{s\}$ ,  $1 \leq s \leq K$  and where  $p(\mathcal{U}, \mathbf{c}) = 0$  whenever  $|\mathcal{U}| \geq 2$  for all  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}$ . Then (3) reduces to the SU scheduling problem considered in [6], [17] which was shown there to be APX-hard. Consequently, we can assert

Table I  
Algorithm I: Algorithm for LTE UL MU-MIMO

- 1: Input  $p(\mathcal{U}, \mathbf{c})$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}$  and  $\mathcal{M}^{\text{narrow}}, \mathcal{M}^{\text{wide}}$
- 2: Determine a feasible allocation over  $\mathcal{M}^{\text{narrow}}$  using Algorithm IIa and let  $\hat{W}^{\text{narrow}}$  denote the corresponding weighted sum rate.
- 3: Determine a feasible allocation over  $\mathcal{M}^{\text{wide}}$  using Algorithm IIb and let  $\hat{W}^{\text{wide}}$  denote the corresponding weighted sum rate.
- 4: Select and output the allocation resulting in  $\hat{W} = \max\{\hat{W}^{\text{narrow}}, \hat{W}^{\text{wide}}\}$ .

Table II  
Algorithm IIa: LRT based module  $\mathcal{M}^{\text{narrow}}$

- 1: Initialize  $p'(\mathcal{U}, \mathbf{c}) \leftarrow p(\mathcal{U}, \mathbf{c})$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$ , stack  $\mathcal{S} = \phi$
- 2: **For**  $j = 1, \dots, N$
- 3: Determine  $(\mathcal{U}^*, \mathbf{c}^*) = \arg \max_{(\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}} \atop \text{Tail}(\mathbf{c})=j} p'(\mathcal{U}, \mathbf{c})$
- 4: **If**  $p'(\mathcal{U}^*, \mathbf{c}^*) > 0$  **Then**
- 5: Set  $\hat{p} = p'(\mathcal{U}^*, \mathbf{c}^*)$  and Push  $(\mathcal{U}^*, \mathbf{c}^*)$  into  $\mathcal{S}$ .
- 6: **For each**  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$  such that  $p'(\mathcal{U}, \mathbf{c}) > 0$
- 7: **If**  $\exists \mathcal{G}_s : \mathcal{U} \cap \mathcal{G}_s \neq \phi \ \& \ \mathcal{U}_j^* \cap \mathcal{G}_s \neq \phi$  or  $\mathbf{c}^* \cap \mathbf{c} \neq \phi$  **Then**
- 8: Update  $p'(\mathcal{U}, \mathbf{c}) \leftarrow p'(\mathcal{U}, \mathbf{c}) - \hat{p}$
- 9: **Else If**  $\exists q \in \mathcal{I} : \alpha^q(\mathcal{U}, \mathbf{c}) = \alpha^q(\mathcal{U}_j^*, \mathbf{c}_j^*) = 1$  **Then**
- 10: Update  $p'(\mathcal{U}, \mathbf{c}) \leftarrow p'(\mathcal{U}, \mathbf{c}) - \hat{p}$
- 11: **Else**
- 12: Update  $p'(\mathcal{U}, \mathbf{c}) \leftarrow p'(\mathcal{U}, \mathbf{c}) - 2\hat{p} \max_{1 \leq q \leq J} \beta^q(\mathcal{U}, \mathbf{c})$ .
- 13: **End If**
- 14: **End For**
- 15: **End If**
- 16: **End For**
- 17: Set stack  $\mathcal{S}' = \phi$
- 18: **While**  $\mathcal{S} \neq \phi$
- 19: Obtain  $(\mathcal{U}, \mathbf{c}) = \text{Pop } \mathcal{S}$
- 20: **If**  $(\mathcal{U}, \mathbf{c}) \cup \mathcal{S}'$  is valid **Then**    %%  $(\mathcal{U}, \mathbf{c}) \cup \mathcal{S}'$  is deemed valid if no user in  $\mathcal{U}$  is incompatible with any user present in  $\mathcal{S}'$  and no chunk in  $\mathcal{S}'$  has an overlap with  $\mathbf{c}$  and all knapsack constraints are satisfied by  $(\mathcal{U}, \mathbf{c}) \cup \mathcal{S}'$ .
- 21: Update  $\mathcal{S}' \leftarrow (\mathcal{U}, \mathbf{c}) \cup \mathcal{S}'$
- 22: **End While**
- 23: Output  $\mathcal{S}'$  and  $\hat{W}^{\text{narrow}} = \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}'} p(\mathcal{U}, \mathbf{c})$ .

that (3) is APX-hard.

Next, consider first Algorithm IIa which outputs a feasible allocation over  $\mathcal{M}^{\text{narrow}}$  yielding a weighted sum rate  $\hat{W}^{\text{narrow}}$ . Let  $\hat{W}^{\text{opt}, \text{narrow}}$  denote the optimal weighted sum rate obtained by solving (3) albeit where all



Table III  
**Algorithm IIb: Greedy module over  $\mathcal{M}^{\text{wide}}$**

- 1: Input  $p(\mathcal{U}, \mathbf{c})$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{wide}}$  and  $\{\mathcal{V}^{(q)}\}_{q=1}^J$ .
- 2: Set  $\mathcal{S} = \phi$  and  $\mathcal{M}' = \mathcal{M}^{\text{wide}}$ .
- 3: **Repeat**
- 4: Determine  $(\mathcal{U}^*, \mathbf{c}^*) = \arg \max_{\substack{(\mathcal{U}, \mathbf{c}) \in \mathcal{M}' \\ \mathcal{S} \cup (\mathcal{U}, \mathbf{c}) \text{ is valid}}} p(\mathcal{U}, \mathbf{c})$ .
- 5: Update  $\mathcal{S} \leftarrow \mathcal{S} \cup (\mathcal{U}^*, \mathbf{c}^*)$  and  $\mathcal{M}' = \mathcal{M}' \setminus \{\mathcal{V}^{(q)} : (\mathcal{U}^*, \mathbf{c}^*) \in \mathcal{V}^{(q)}\}$
- 6: **Until**  $(\mathcal{U}^*, \mathbf{c}^*) = \phi$  or  $\mathcal{M}' = \phi$ .
- 7: Output  $\mathcal{S}$  and  $\hat{W}^{\text{wide}} = \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}} p(\mathcal{U}, \mathbf{c})$ .

pairs  $(\mathcal{U}, \mathbf{c})$  are restricted to lie in  $\mathcal{M}^{\text{narrow}}$ . We will prove that

$$\hat{W}^{\text{narrow}} \geq \frac{\hat{W}^{\text{opt, narrow}}}{1 + T + \Delta + 2J}. \quad (25)$$

We present a proof that invokes notation and results developed for LRT based SU scheduling in [6] as much as possible, and highlights mainly the key differences. These differences are novel and crucial since they allow us to co-schedule multiple users on a chunk while respecting incompatibility constraints and to satisfy multiple knapsack constraints.

Note that Algorithm IIa builds up the stack  $\mathcal{S}$  in  $N$  steps. In particular let  $S_j$ ,  $j = 1, \dots, N$  be the element that is added in the  $j^{\text{th}}$  step and note that either  $S_j = \phi$  or it is equal to some pair  $(\mathcal{U}_j^*, \mathbf{c}_j^*)$ . As in [6], we use two functions  $p_1^{(j)} : \mathcal{M}^{\text{narrow}} \rightarrow \mathbb{R}_+$  and  $p_2^{(j)} : \mathcal{M}^{\text{narrow}} \rightarrow \mathbb{R}_+$  for  $j = 0, \dots, N$  to track the function  $p'(\cdot)$  as the stack  $\mathcal{S}$  is being built up over  $N$  steps and in particular we set  $p_1^{(0)}(\mathcal{U}, \mathbf{c}) = 0$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$  and  $p_2^{(0)}(\mathcal{U}, \mathbf{c}) = p(\mathcal{U}, \mathbf{c})$ ,  $\forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$ . For our problem at hand, we define  $\{p_1^{(j)}(\mathcal{U}, \mathbf{c}), p_2^{(j)}(\mathcal{U}, \mathbf{c})\}$  recursively as

$$p_1^{(j)}(\mathcal{U}, \mathbf{c}) = \begin{cases} (p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*))^+ \mathcal{X}(p_2^{(j-1)}(\mathcal{U}, \mathbf{c}) > 0), & \text{If } \mathbf{c}_j^* \cap \mathbf{c} \neq \phi \\ (p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*))^+ \mathcal{X}(p_2^{(j-1)}(\mathcal{U}, \mathbf{c}) > 0), & \text{ElseIf } \exists \mathcal{G}_s : \mathcal{U} \cap \mathcal{G}_s \neq \phi \ \& \ \mathcal{U}_j^* \cap \mathcal{G}_s \neq \phi \\ (p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*))^+ \mathcal{X}(p_2^{(j-1)}(\mathcal{U}, \mathbf{c}) > 0), & \text{ElseIf } \exists q \in \mathcal{I} : \alpha^q(\mathcal{U}, \mathbf{c}) = \alpha^q(\mathcal{U}_j^*, \mathbf{c}_j^*) = 1 \\ 2(p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*))^+ \mathcal{X}(p_2^{(j-1)}(\mathcal{U}, \mathbf{c}) > 0) \max_{1 \leq q \leq J} \beta^q(\mathcal{U}, \mathbf{c}), & \text{Otherwise} \end{cases}$$

$$p_2^{(j)}(\mathcal{U}, \mathbf{c}) = p_2^{(j-1)}(\mathcal{U}, \mathbf{c}) - p_1^{(j)}(\mathcal{U}, \mathbf{c}), \quad (26)$$

where  $(x)^+ = \max\{x, 0\}$ ,  $x \in \mathbb{R}$ ,  $\mathcal{X}(\cdot)$  denotes the indicator function and  $(\mathcal{U}_j^*, \mathbf{c}_j^*) =$

$\arg \max_{(\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}_{\text{Tail}(\mathbf{c})=j}} p_2^{(j-1)}(\mathcal{U}, \mathbf{c})$ . Hence, we have that

$$p_2^{(j-1)}(\mathcal{U}, \mathbf{c}) = p_2^{(j)}(\mathcal{U}, \mathbf{c}) + p_1^{(j)}(\mathcal{U}, \mathbf{c}), \quad \forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}, j = 1, \dots, N. \quad (27)$$

It can be noted that

$$\begin{aligned} p_2^{(j)}(\mathcal{U}, \mathbf{c}) &\leq 0, \quad \forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}} : \text{Tail}(\mathbf{c}) \leq j \\ p_2^{(k)}(\mathcal{U}, \mathbf{c}) &\leq p_2^{(j)}(\mathcal{U}, \mathbf{c}), \quad \forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}} \text{ \& } k \geq j. \end{aligned} \quad (28)$$

Further, to track the stack  $\mathcal{S}'$  which is built in the while loop of the algorithm, we define stacks  $\{\mathcal{S}_j^*\}_{j=0}^N$  where  $\mathcal{S}_N^* = \phi$  and  $\mathcal{S}_j^*$  is the value of  $\mathcal{S}'$  after the Algorithm has tried to add  $\cup_{m=j+1}^N S_m$  to  $\mathcal{S}'$  (starting from  $\mathcal{S}' = \phi$ ) so that  $\mathcal{S}_0^*$  is the stack  $\mathcal{S}'$  that is the output of the Algorithm. Note that  $\mathcal{S}_{j+1}^* \subseteq \mathcal{S}_j^* \subseteq \mathcal{S}_{j+1}^* \cup S_{j+1}$ . Next, for  $j = 0, \dots, N$ , we let  $W^{(j) \text{ opt}}$  denote the optimal solution to (3) but where  $\mathcal{M}$  is replaced by  $\mathcal{M}^{\text{narrow}}$  and the function  $p(\cdot)$  is replaced by  $p_2^{(j)}(\cdot)$ . Further, let  $W^{(j)} = \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}_j^*} p_2^{(j)}(\mathcal{U}, \mathbf{c})$  and note that  $\hat{W}^{\text{opt}, \text{narrow}} = W^{(0) \text{ opt}}$  and  $\hat{W}^{\text{narrow}} = W^{(0)}$ . We will show via induction that

$$W^{(j) \text{ opt}} \leq (T + 1 + \Delta + 2J)W^{(j)}, \quad \forall j = N, \dots, 0, \quad (29)$$

which includes the claim in (25) at  $j = 0$ . The base case  $W^{(N) \text{ opt}} \leq (T + 1 + \Delta + 2J)W^{(N)}$  is readily true since  $\mathcal{S}_N^* = \phi$  and  $p_2^{(N)}(\mathcal{U}, \mathbf{c}) \leq 0, \quad \forall (\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$ . Assume that (29) holds for some  $j$ . We focus only on the main case in which  $S_j = (\mathcal{U}_j^*, \mathbf{c}_j^*) \neq \phi$  (the remaining case holds trivially true). Note that since  $(\mathcal{U}_j^*, \mathbf{c}_j^*)$  is added to the stack  $\mathcal{S}$  in the algorithm,  $p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*) > 0$ . Then from the update formulas (26), we must have that  $p_2^{(j)}(\mathcal{U}_j^*, \mathbf{c}_j^*) = 0$ . Using the fact that  $\mathcal{S}_{j-1}^* \subseteq \mathcal{S}_j^* \cup (\mathcal{U}_j^*, \mathbf{c}_j^*)$  together with the induction hypothesis, we can conclude that

$$W^{(j)} = \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}_j^*} p_2^{(j)}(\mathcal{U}, \mathbf{c}) = \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}_{j-1}^*} p_2^{(j)}(\mathcal{U}, \mathbf{c}) \geq \frac{W^{(j) \text{ opt}}}{T + 1 + \Delta + 2J}. \quad (30)$$

Next, we will show that

$$\sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}_{j-1}^*} p_1^{(j)}(\mathcal{U}, \mathbf{c}) \geq p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*). \quad (31)$$

Towards this end, suppose that  $\mathcal{S}_{j-1}^* = \mathcal{S}_j^* \cup (\mathcal{U}_j^*, \mathbf{c}_j^*)$ . Then, recalling (26) we can deduce that (31) is true since  $p_1^{(j)}(\mathcal{U}_j^*, \mathbf{c}_j^*) = p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*)$ . Suppose now that  $\mathcal{S}_{j-1}^* = \mathcal{S}_j^*$ . In this case we can have two possibilities. In the first one  $(\mathcal{U}_j^*, \mathbf{c}_j^*)$  cannot not be added to  $\mathcal{S}_j^*$  due to the presence of a pair  $(\mathcal{U}', \mathbf{c}') \in \mathcal{S}_j^*$  for which

at-least one of these three conditions are satisfied:  $\exists \mathcal{G}_s : \mathcal{U}' \cap \mathcal{G}_s \neq \phi \ \& \ \mathcal{U}_j^* \cap \mathcal{G}_s \neq \phi; \ \mathbf{c}' \cap \mathbf{c}_j^* \neq \phi$  and  $\exists q \in \mathcal{I} : \alpha^q(\mathcal{U}', \mathbf{c}') = \alpha^q(\mathcal{U}_j^*, \mathbf{c}_j^*)=1$ . Since any pair  $(\mathcal{U}', \mathbf{c}') \in \mathcal{S}_j^*$  was added to  $\mathcal{S}$  in the algorithm after the  $j^{th}$  step, from the second inequality in (28) we must have that  $p_2^{(j-1)}(\mathcal{U}', \mathbf{c}') > 0$ . Recalling (26) we can then deduce that  $p_1^{(j)}(\mathcal{U}', \mathbf{c}') = p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*)$  which proves (31). In the second possibility,  $(\mathcal{U}_j^*, \mathbf{c}_j^*)$  cannot not be added to  $\mathcal{S}_j^*$  due to a generic knapsack constraint being violated. In other words, for some  $q \in \{1, \dots, J\}$ , we have that

$$\sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}_j^*} \beta^q(\mathcal{U}, \mathbf{c}) > 1 - \beta^q(\mathcal{U}_j^*, \mathbf{c}_j^*). \quad (32)$$

Since  $(\mathcal{U}_j^*, \mathbf{c}_j^*) \in \mathcal{M}^{\text{narrow}}$ ,  $\beta^q(\mathcal{U}_j^*, \mathbf{c}_j^*) \leq 1/2$  so that

$$2 \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}_j^*} \max_{1 \leq q \leq J} \beta^q(\mathcal{U}, \mathbf{c}) \geq 2 \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}_j^*} \beta^q(\mathcal{U}, \mathbf{c}) > 1. \quad (33)$$

which along with (26) also proves (31). Thus, we have established the claim in (31).

Finally, letting  $V^{(j) \text{ opt}}$  denote the optimal solution to (3) but where  $\mathcal{M}$  is replaced by  $\mathcal{M}^{\text{narrow}}$  and the function  $p(\cdot)$  is replaced by  $p_1^{(j)}(\cdot)$ , we will show that

$$p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*) \geq \frac{V^{(j) \text{ opt}}}{T + 1 + \Delta + 2J}. \quad (34)$$

Towards this end, from (26) we note that for any pair  $(\mathcal{U}, \mathbf{c}) \in \mathcal{M}^{\text{narrow}}$ ,  $p_1^{(j)}(\mathcal{U}, \mathbf{c}) \leq p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*)$ . Let  $\mathcal{V}_1^{(j) \text{ opt}}$  be an optimal allocation of pairs that results in  $V^{(j) \text{ opt}}$ . For any two pairs  $(\mathcal{U}_1, \mathbf{c}_1), (\mathcal{U}_2, \mathbf{c}_2) \in \mathcal{V}_1^{(j) \text{ opt}}$  we must have that for each  $\mathcal{G}_s$   $1 \leq s \leq L$ , at-least one of  $\mathcal{U}_1 \cap \mathcal{G}_s$  and  $\mathcal{U}_2 \cap \mathcal{G}_s$  is  $\phi$ , as well as  $\mathbf{c}_1 \cap \mathbf{c}_2 = \phi$ . In addition,  $|\mathcal{U}_1|$  and  $|\mathcal{U}_2|$  are no greater than  $T$ . Thus we can have at-most  $T$  such pairs  $\{(\mathcal{U}, \mathbf{c})\}$  in  $\mathcal{V}_1^{(j) \text{ opt}}$  for which  $\exists \mathcal{G}_s : \mathcal{U} \cap \mathcal{G}_s \neq \phi \ \& \ \mathcal{U}_j^* \cap \mathcal{G}_s \neq \phi$ . Further, using the first inequality in (28) we see that any pair  $(\mathcal{U}, \mathbf{c})$  for which  $\mathbf{c} \cap \mathbf{c}_j^* \neq \phi$  and  $p_1^{(j)}(\mathcal{U}, \mathbf{c}) = p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*)$  must have  $\text{Tail}(\mathbf{c}) \geq j$  so that  $j \in \mathbf{c}$ . Thus,  $\mathcal{V}_1^{(j) \text{ opt}}$  can include at-most one pair  $(\mathcal{U}, \mathbf{c})$  for which  $\mathbf{c} \cap \mathbf{c}_j^* \neq \phi$ . Next, there can be at-most  $\Delta$  constraints in  $\mathcal{I}$  for which  $\alpha^q(\mathcal{U}_j^*, \mathbf{c}_j^*) = 1, q \in \mathcal{I}$  is satisfied. For each such constraint  $q \in \mathcal{I}$  we can pick at-most one pair  $(\mathcal{U}, \mathbf{c})$  for which  $\alpha^q(\mathcal{U}, \mathbf{c}) = 1$  and  $p_1^{(j)}(\mathcal{U}, \mathbf{c}) = p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*)$ . Thus,  $\mathcal{V}_1^{(j) \text{ opt}}$  can include at-most  $\Delta$  such pairs, one for each constraint. Now the remaining pairs in  $\mathcal{V}_1^{(j) \text{ opt}}$  (whose users do not intersect  $\mathcal{U}_j^*$  and whose chunks do not intersect  $\mathbf{c}_j^*$  and which do not violate any binary knapsack constraint in the presence of  $(\mathcal{U}_j^*, \mathbf{c}_j^*)$ ) must satisfy the generic knapsack constraints. Let these pairs form the set  $\tilde{\mathcal{V}}_1^{(j) \text{ opt}}$  so that

$$\sum_{(\mathcal{U}, \mathbf{c}) \in \tilde{\mathcal{V}}_1^{(j) \text{ opt}}} p_1^{(j)}(\mathcal{U}, \mathbf{c}) = \sum_{(\mathcal{U}, \mathbf{c}) \in \tilde{\mathcal{V}}_1^{(j) \text{ opt}}} 2p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*) \max_{1 \leq q \leq J} \beta^q(\mathcal{U}, \mathbf{c}) \leq 2p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*) \sum_{q=1}^J \sum_{(\mathcal{U}, \mathbf{c}) \in \tilde{\mathcal{V}}_1^{(j) \text{ opt}}} \beta^q(\mathcal{U}, \mathbf{c})$$

$$\leq 2Jp_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*).$$

Combining these observations we have that

$$V^{(j) \text{ opt}} = \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{V}_1^{(j) \text{ opt}}} p_1^{(j)}(\mathcal{U}, \mathbf{c}) \leq (1 + T + \Delta + 2J)p_2^{(j-1)}(\mathcal{U}_j^*, \mathbf{c}_j^*), \quad (35)$$

which is the desired result in (34).

Thus, using (30), (31) and (34) we can conclude that

$$(1 + T + \Delta + 2J) \sum_{(\mathcal{U}, \mathbf{c}) \in \mathcal{S}_{j-1}^*} \underbrace{(p_1^{(j)}(\mathcal{U}, \mathbf{c}) + p_2^{(j)}(\mathcal{U}, \mathbf{c}))}_{p_2^{(j-1)}(\mathcal{U}, \mathbf{c})} \geq V^{(j) \text{ opt}} + W^{(j) \text{ opt}} \geq W^{(j-1) \text{ opt}}. \quad (36)$$

which proves the induction step and proves the claim in (25).

Let us now consider the remaining part which arises when  $\mathcal{M}^{\text{wide}} \neq \emptyset$ . Consider first Algorithm IIb which outputs a feasible allocation over  $\mathcal{M}^{\text{wide}}$  yielding a weighted sum rate  $\hat{W}^{\text{wide}}$ . Let  $\hat{W}^{\text{opt, wide}}$  denote the optimal weighted sum rate obtained by solving (3) albeit where all pairs  $(\mathcal{U}, \mathbf{c})$  are restricted to lie in  $\mathcal{M}^{\text{wide}}$ . We will prove that

$$\hat{W}^{\text{wide}} \geq \frac{\hat{W}^{\text{opt, wide}}}{J}. \quad (37)$$

Let  $\mathcal{V}^{\text{opt, wide}}$  be an optimal allocation of pairs from  $\mathcal{M}^{\text{wide}}$  that results in a weighted sum rate  $\hat{W}^{\text{opt, wide}}$ . Clearly, in order to meet the knapsack constraints,  $\mathcal{V}^{\text{opt, wide}}$  can include at-most one pair from each  $\mathcal{V}^{(q)}$ ,  $1 \leq q \leq J$  so that there can be at-most  $J$  pairs in  $\mathcal{V}^{\text{opt, wide}}$ . Thus, by selecting the pair yielding the maximum weighted sum-rate we can achieve at-least  $\hat{W}^{\text{opt, wide}}/J$ . The greedy algorithm first selects the pair yielding the maximum weighted sum rate among all pairs in  $\mathcal{M}^{\text{wide}}$  and then attempts to add pairs to monotonically improve the objective. Thus, we can conclude that (37) must be true..

Notice that we select  $\hat{W} = \max\{\hat{W}^{\text{narrow}}, \hat{W}^{\text{wide}}\}$  so that

$$\hat{W} \geq \max \left\{ \frac{\hat{W}^{\text{opt, narrow}}}{1 + T + \Delta + 2J}, \frac{\hat{W}^{\text{opt, wide}}}{J} \right\}. \quad (38)$$

It is readily seen that

$$\hat{W}^{\text{opt}} \leq \hat{W}^{\text{opt, narrow}} + \hat{W}^{\text{opt, wide}}. \quad (39)$$

(38) and (39) together prove the theorem.

## APPENDIX B

## PROOF OF PROPOSITION 2

*Proof:* On any RB  $j$ , consider any fixed pair  $\mathcal{U} = \{u, v\} \subseteq \{1, \dots, K\}$  and define the set function

$$g(\mathcal{A}) = p^{\text{su}}(\mathcal{U} \cap \mathcal{A}, j) \mathcal{X}(|\mathcal{U} \cap \mathcal{A}| = 1) + p^{\text{sic}}(\mathcal{U}, j) \mathcal{X}(|\mathcal{U} \cap \mathcal{A}| = 2), \forall \mathcal{A} \subseteq \{1, \dots, K\}. \quad (40)$$

Our first aim is to prove that  $g(\cdot)$  defined above is a monotonic sub-modular set function. First, note that the weighted sum rate in (18) can also be written as,

$$p^{\text{sic}}(\mathcal{U}, j) = (w_{\hat{u}} - w_{\hat{v}}) p^{\text{su}}(\hat{u}, j) + w_{\hat{v}} \log \left| \mathbf{I} + \mathbf{h}_{\hat{v},j} \mathbf{h}_{\hat{v},j}^{\dagger} + \mathbf{h}_{\hat{u},j} \mathbf{h}_{\hat{u},j}^{\dagger} \right| \quad (41)$$

$$\geq p^{\text{su}}(\hat{v}, j) + w_{\hat{u}} \log(1 + \mathbf{h}_{\hat{u},j}^{\dagger} (\mathbf{I} + \mathbf{h}_{\hat{v},j} \mathbf{h}_{\hat{v},j}^{\dagger})^{-1} \mathbf{h}_{\hat{u},j}) \quad (42)$$

so that  $p^{\text{sic}}(\mathcal{U}, j) \geq \max\{p^{\text{su}}(\hat{u}, j), p^{\text{su}}(\hat{v}, j)\}$ , which suffices to prove the monotonicity of  $g(\cdot)$ . Then, to prove sub-modularity we must show that,

$$g(\mathcal{A} \cup \{q\}) - g(\mathcal{A}) \geq g(\mathcal{B} \cup \{q\}) - g(\mathcal{B}), \forall \mathcal{A} \subseteq \mathcal{B} \subseteq \{1, \dots, K\} \text{ \& } q \in \{1, \dots, K\} \setminus \mathcal{B}. \quad (43)$$

To prove (43) we consider any  $\mathcal{A} \subseteq \mathcal{B} \subseteq \{1, \dots, K\}$  so that  $\mathcal{A} \cap \mathcal{U} \subseteq \mathcal{B} \cap \mathcal{U}$  and consider the following cases. First consider the case,  $|\mathcal{A} \cap \mathcal{U}| = |\mathcal{B} \cap \mathcal{U}|$  which implies that both  $\mathcal{A}, \mathcal{B}$  contain the same user(s) from  $\mathcal{U}$  so that (43) must hold with equality. Then, suppose  $|\mathcal{A} \cap \mathcal{U}| < |\mathcal{B} \cap \mathcal{U}|$ . In this case, upon exploiting the inequality

$$p^{\text{sic}}(\mathcal{U}, j) \leq p^{\text{su}}(u, j) + p^{\text{su}}(v, j), \quad (44)$$

together with the fact that  $g(\mathcal{B} \cup \{q\}) - g(\mathcal{B}) = 0$  when  $|\mathcal{B} \cap \mathcal{U}| = 2$ , we can conclude that (43) must hold. Then, since the set function  $h(\cdot)$  in (23) is a linear combination of  $NK(K+1)/2$  monotonic sub-modular set functions in which the combining coefficients are all positive, we can assert that it must be a monotonic sub-modular set function as well.

## APPENDIX C

## APPENDIX: MODELING 3GPP LTE CONTROL CHANNEL CONSTRAINTS

Note that by placing restrictions on the location where a particular user's control packet can be sent and the size of that packet, the system can reduce the number of blind decoding attempts that have to be made by that user in order to receive its control packet. We note that a user is unaware of whether there is a control packet intended for it and consequently must check all possible locations where its control packet could be present

assuming each possible packet size. Each control packet carries a CRC bit sequence scrambled using the unique user identifier which helps the user deduce whether the examined packet is meant for it [1]. In the 3GPP LTE system, the minimum allocation unit in the downlink control channel is referred to as the control channel element (CCE). Let  $\{1, \dots, R\}$  be a set of CCEs available for conveying UL grants. A contiguous chunk of CCEs from  $\{1, \dots, R\}$  that can be assigned to a user is referred to as a PDCCH. The size of each PDCCH is referred to as an aggregation level and must belong to the set  $\{1, 2, 4, 8\}$ . Let  $\mathcal{D}$  denote the set of all possible such PDCCHs. For each user the BS first decides an aggregation level, based on its average (long-term) SINR. Then, using that users' unique identifier (ID) together with its aggregation level, the BS obtains a small subset of non-overlapping PDCCHs from  $\mathcal{D}$  (of cardinality no greater than 6) that are eligible to be assigned to that user. Let  $\mathcal{D}_u$  denote this subset of eligible PDCCHs for a user  $u$ . Then, if user  $u$  is scheduled only one PDCCH from  $\mathcal{D}_u$  must be assigned to it, i.e., must be used to convey its UL grant. Note that while the PDCCHs that belong to the eligible set of any one user are non-overlapping, those that belong to eligible sets of any two different users can overlap. As a result, the BS scheduler must also enforce the constraint that two PDCCHs that are assigned to two different scheduled users, respectively, must not overlap.

Next, the constraint that each scheduled user can be assigned only one PDCCH from its set of eligible PDCCHs can be enforced as follows. First, define a set  $\mathcal{V}_u$  containing  $|\mathcal{D}_u|$  *virtual users* for each user  $u$ ,  $1 \leq u \leq K$ , where each virtual user in  $\mathcal{V}_u$  is associated with a unique PDCCH in  $\mathcal{D}_u$  and all the parameters (such as uplink channels, queue size etc.) corresponding to each virtual user in  $\mathcal{V}_u$  are identical to those of user  $u$ . Let  $\tilde{\mathcal{U}}$  be the set of all possible subsets of such virtual users, such that each subset has a cardinality no greater than  $T$  and contains no more than one virtual user corresponding to the same user. Defining  $\tilde{\mathcal{M}} = \tilde{\mathcal{U}} \times \mathcal{C}$ , we can then pose (3) over  $\tilde{\mathcal{M}}$  after setting  $L = K$  with  $\mathcal{G}_s = \mathcal{V}_s$ ,  $1 \leq s \leq K$ . Consequently, by defining the virtual users corresponding to each user as being mutually incompatible, we have enforced the constraint that at-most one virtual user for each user can be selected, which in turn is equivalent to enforcing that each scheduled user can be assigned only one PDCCH from its set of eligible PDCCHs.

Finally, consider the set of all eligible PDCCHs,  $\{\mathcal{D}_u\}_{u=1}^K$ . Note that this set is decided by the set of active users and their long-term SINRs. Recall that each PDCCH in  $\{\mathcal{D}_u\}_{u=1}^K$  maps to a unique virtual user. To ensure that PDCCHs that are assigned to two virtual users corresponding to two different users do not overlap, we can define multiple binary knapsack constraints. Clearly  $R$  such knapsack constraints suffice (indeed can be much more than needed), where each constraint corresponds to one CCE and has a weight of one for every pair  $(\tilde{\mathcal{U}}, c) \in \tilde{\mathcal{M}}$  wherein  $\tilde{\mathcal{U}}$  contains a virtual user corresponding to a PDCCH which includes that CCE. Then, a

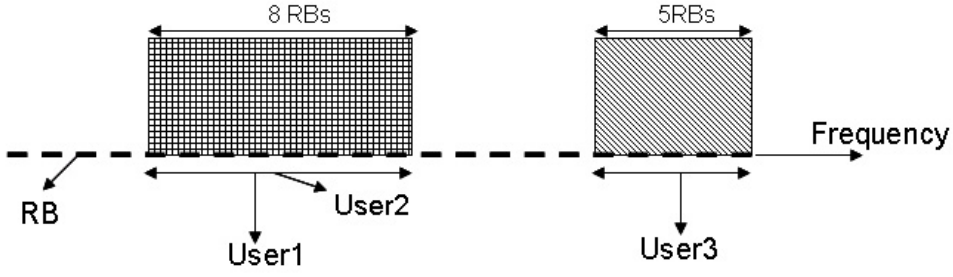


Figure 1. A Feasible RB Allocation in the LTE UL

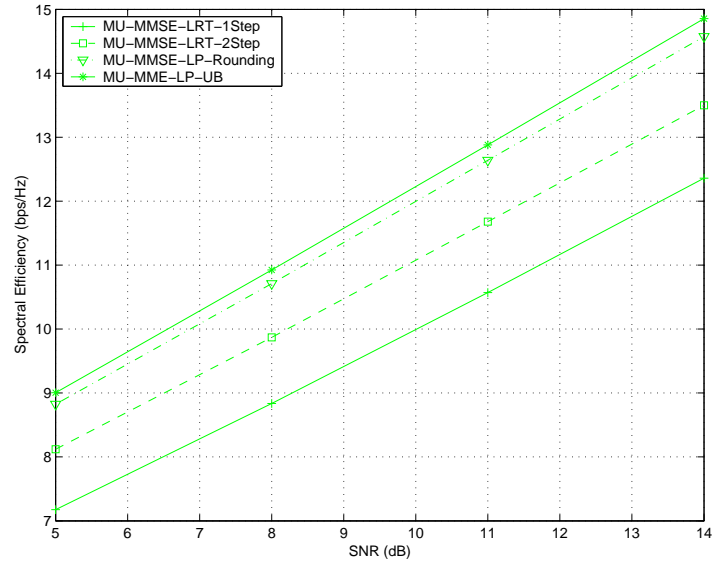


Figure 2. Average spectral efficiency versus SNR (dB): MU Scheduling with MMSE receiver.

useful consequence of the fact that in LTE the set  $\mathcal{D}_u$  for each user  $u$  is extracted from  $\mathcal{D}$  via a well designed hash function (which accepts each user's unique ID as input), is that these resulting knapsack constraints are *column-sparse*.



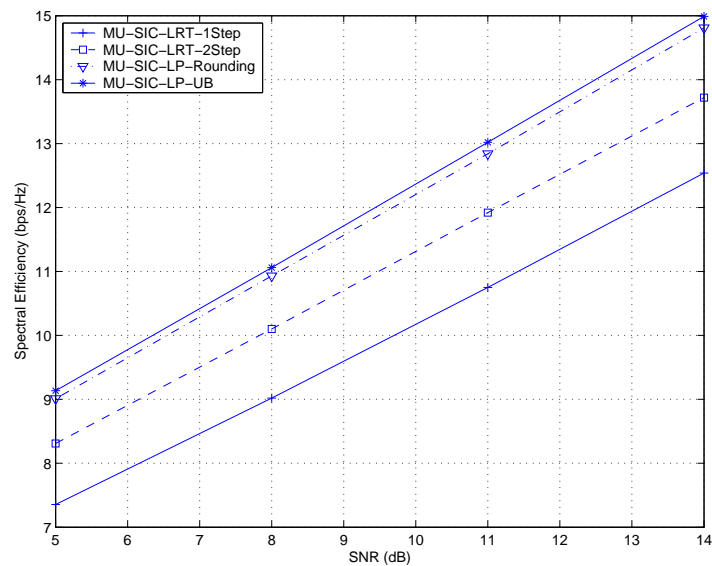


Figure 3. Average spectral efficiency versus SNR (dB): MU Scheduling with SIC receiver.

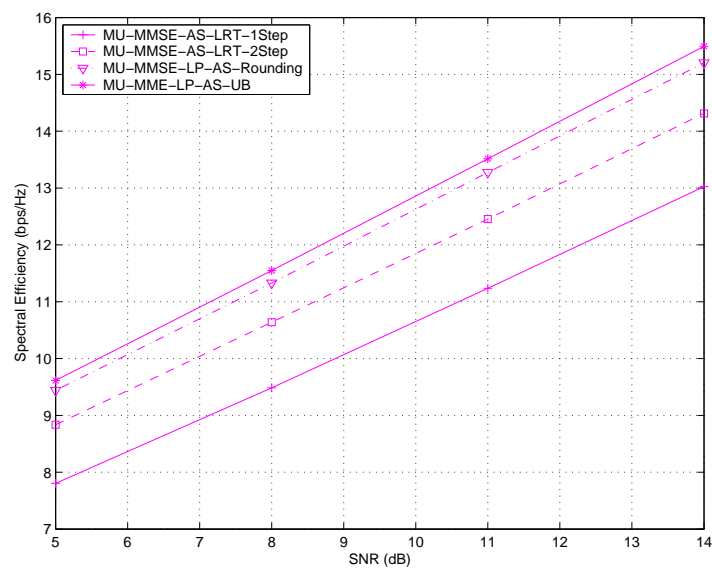


Figure 4. Average spectral efficiency versus SNR (dB): MU Scheduling with MMSE and Antenna Selection.

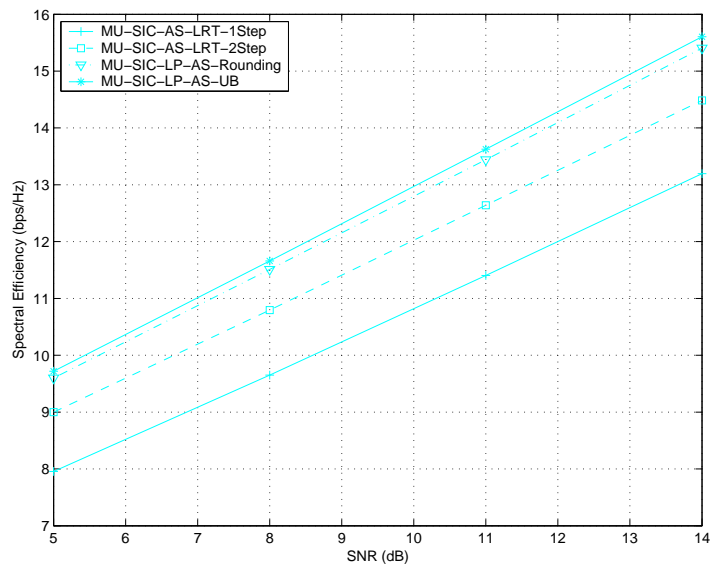


Figure 5. Average spectral efficiency versus SNR (dB): MU Scheduling with SIC and Antenna Selection.

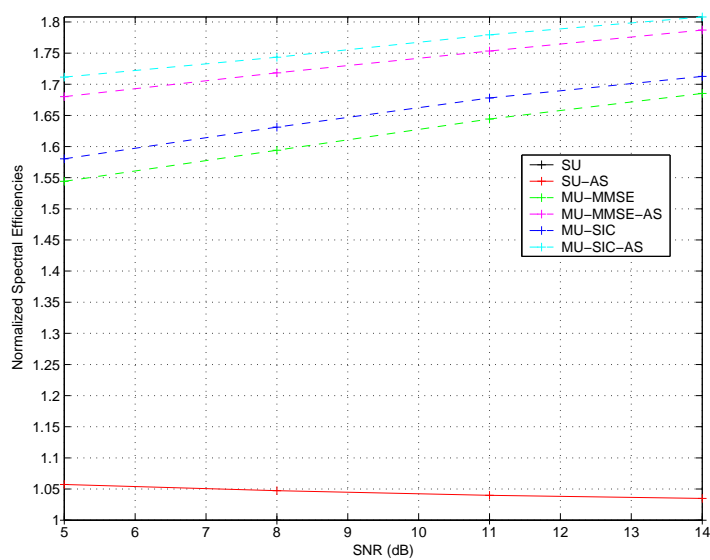


Figure 6. Normalized spectral efficiency versus SNR (dB)

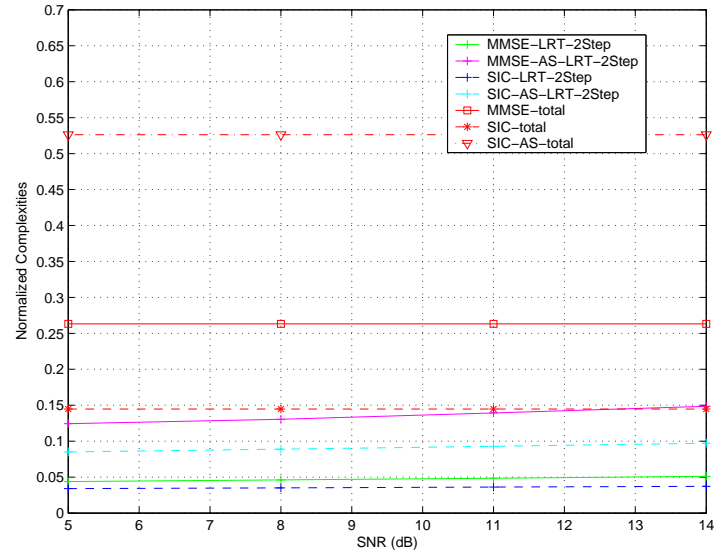


Figure 7. Normalized complexity versus SNR (dB)

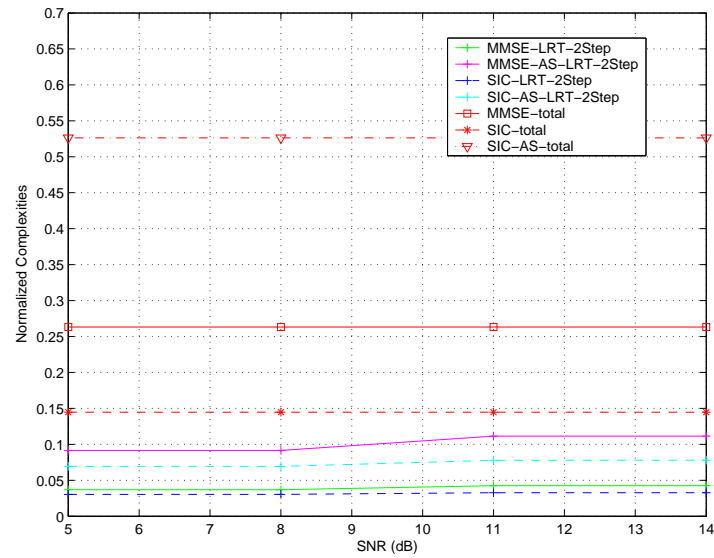


Figure 8. Normalized complexity versus SNR (dB). Chunk sizes are included in complexity computations.

Parameter	Assumption
Deployment scenario	IMT Urban Micro (UMi)
Duplex method and bandwidth	FDD: 10MHz for uplink
Cell layout	Hex grid 19 sites, 3 cells/site
Transmission power at user	23 dBm
Average number of users per sector	10 or 15
Network synchronization	Synchronized
Antenna configuration (eNB)	4 RX co-polarized ant., $0.5\text{-}\lambda$ spacing
Antenna configuration (user)	1 TX ant.
Uplink transmission scheme	Dynamic MU scheduling, MU pairing: Max 2/RB users aligned pairing;
Fractional power control	$P_0 = -85$ dB, $\alpha = 0.8$
Uplink scheduler	PF in time and frequency
Scheduling granularity:	1 RB
CQI assumptions	1 ms periodicity and 7 ms delay CQI without errors.
Uplink HARQ scheme	Synchronous, non-adaptive Chase Combining
Uplink receiver type	MMSE-IRC and SIC-IRC
Channel estimation error	NA

Table IV  
PARAMETERS FOR SYSTEM LEVEL SIMULATIONS

Scheduling method	cell average	5% cell-edge
LRT SU	1.6214	0.0655
LRT MU with MMSE	1.9246 (18.70%)	0.0524
LRT MU with SIC	2.0651 (27.37%)	0.0745
LRT-Sequential MU with MMSE	1.8196 (12.22%)	0.0627
LRT-Sequential MU with SIC	1.9537 (20.5%)	0.0665

Table V  
SPECTRAL EFFICIENCY OF LRT BASED SU AND MU UL SCHEDULING SCHEMES. AN AVERAGE OF 10 USERS ARE PRESENT IN EACH CELL AND ALL ASSOCIATED ACTIVE USERS CAN BE SCHEDULED IN EACH INTERVAL.

LRT-MU scheduling with:	cell average	5% cell-edge
Knapsack constraint	1.7833	0.0266
pre-selection 1	1.7940 (0.6%)	0.0419 (57.52%)
pre-selection 2	1.7908 (0.4%)	0.0414 (55.64%)
pre-selection 3	1.8265 (2.42%)	0.0444 (66.92%)

Table VI  
SPECTRAL EFFICIENCY OF MU UL SCHEDULING SCHEMES WITH MMSE RECEIVER. AN AVERAGE OF 15 USERS ARE PRESENT IN EACH CELL AND AT-MOST 7 FIRST-TRANSMISSION USERS CAN BE SCHEDULED IN EACH INTERVAL.

LRT-MU scheduling with:	cell average	5% cell-edge
Knapsack constraint	1.8865	0.0411
pre-selection 1	2.0082 (6.45%)	0.0527 (28.22%)
pre-selection 2	1.8980 (0.61%)	0.0451 (9.73%)
pre-selection 4	2.1069 (11.68%)	0.0531 (29.2%)

Table VII

SPECTRAL EFFICIENCY OF MU UL SCHEDULING SCHEMES WITH SIC RECEIVER. AN AVERAGE OF 15 USERS ARE PRESENT IN EACH CELL AND AT-MOST 7 FIRST-TRANSMISSION USERS CAN BE SCHEDULED IN EACH INTERVAL.

